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On n-point amplitudes in $N = 4$ SYM

A. Mironov[ab] **and A. Morozov**[b]
[a] *Lebedev Physics Institute, Moscow, Russia*
[b] *ITEP, Moscow, Russia*
E-mail: mironov@itep.ru, morozov@itep.ru

T.N. Tomaras
Department of Physics and Institute of Plasma Pysics, University of Crete, Heraklion, Greece
E-mail: tomaras@physics.uoc.gr

Abstract: The computation of n-point planar amplitudes in N=4 SYM at strong coupling is known to be reduced to the search for solutions of the integrable 2d SO(4,2) σ-model with growing asymptotics on the world-sheet and to the study of their Whitham deformations induced by an ϵ-regularization, which breaks both integrability and SO(4,2) symmetry. A multi-parameter (moduli) family of such solutions is constructed for $n = 4$. They all correspond to the same s and t and some are related by SO(4,2) transformations. Nevertheless, they lead to different regularized areas, whose minimum is the Alday-Maldacena solution. A brief review of results on n-point amplitudes is also provided, with special emphasis on the underlying equivalence of the above regularized minimal area in AdS and a double contour integral along the same boundary, two purely geometric quantities.

Keywords: AdS-CFT Correspondence, Gauge-gravity correspondence, Supersymmetric gauge theory.

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1. Introduction and results

A new significant step was taken recently [1] towards the computation of planar $N = 4$ Super-Yang-Mills (SYM) n-point amplitudes beyond ordinary perturbation theory, using the gauge/string correspondence. The celebrated exponentiation BDS-hypothesis of [2] was verified at strong coupling for the four point amplitude. To achieve this, the authors made several ingenious choices and educated guesses in order to deal with the difficulties in the string side of the AdS/CFT duality [3, 4], studied also in a number of recent publications [5]–[16].

In particular, in [1]

(i) the σ-model action instead of the Nambu-Goto one was used, what allowed to perform a T-duality *a la* [17],

(ii) a minimal surface was constructed for $n = 4$ based on previous considerations [18],

(iii) a rather unusual dimensional regularization was employed, instead of, for instance, the one described in [19],

(iv) a skilful handling of the resulting integrals was required, and finally

(v) the KLOV interpolation [20] between weak and strong coupling regimes was used.

A better understanding of these and related issues seems to be an unavoidable step, before one can generalize this method and derive the dilogarithmic BDS formula [2] for $n > 4$. Before we describe the little progress made in the present paper, let us briefly summarize the basic ingredients of the AdS/CFT correspondence.

1.1 Gauge theory side

Start with the $N = 4$ SYM.

(a) According to the conjecture of [2], the *planar* (at least, maximally helicity violating) n-point amplitude $\mathcal{A}_n$ in $N = 4$ *SYM* gauge theory has the form

$$\mathcal{A}_n = \mathcal{A}_{n,tree} \times \mathcal{M}_n \tag{1.1}$$

where $\mathcal{M}_n$ does not depend on any color and helicity factors. In the supersymmetric case, the leading logarithm approximation (LLA) [21] is not required for the amplitudes to exponentiate: both the infrared divergent and the finite parts of amplitudes are surprisingly simple exponentials. Explicitly, its infrared divergent part is the universal exponential

$$\mathcal{M}_{IR} \sim \exp\left(-\frac{1}{4}\sum_{l=1}^{\infty}\lambda^l\left[\gamma_{(l)} + 2l\epsilon g^{(l)}\right] I_n^{(1)}(l\epsilon)\right) \tag{1.2}$$

with the 1-loop scalar box integral

$$I_n^{(1)}(\epsilon) = \frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{s_{i,i+1}}\right)^{\epsilon} \tag{1.3}$$

$s_{i,i+1}$ is the square of the sum of the i and $i+1$ external momenta and the function $\gamma(\lambda) \equiv \sum_l \gamma_{(l)}\lambda^l$ of the 't Hooft coupling $\lambda \equiv \frac{N_c\alpha_S}{2\pi}\left(4\pi e^{\Gamma'(1)}\right)^{\epsilon}$ is called soft or cusp (Wilson line [22, 23]) anomalous dimension [24]. μ and ϵ are the standard dimensional regularization parameters.

(b) The finite part of $\mathcal{M}_n$ is also expressed through its 1-loop counterpart $F_n^{(1)}$ as:

$$F_n = \exp\left(\frac{1}{4}\gamma(\lambda)F_n^{(1)} + C(\lambda)\right) \tag{1.4}$$

where $C(\lambda)$ depends neither on n, nor on kinematics. If this is true, then the main non-trivial quantity entering both (1.2) and (1.4) is $\gamma(\lambda)$, the anomalous dimension of twist-two operators, an eigenvalue of a not yet fully known Bethe ansatz [25], but with known strong 't Hooft coupling asymptotics $\gamma(\lambda) \sim \sqrt{\lambda} + \text{const} + O(1/\sqrt{\lambda})$, [26, 18, 27, 28].

(c) The 1-loop amplitude $F_n^{(1)}$ may be expressed as a sum over 4-clusters in an auxiliary polygon Π, figures 4), (2, formed by the external momenta $\mathbf{p}_a$ of the process at hand, which plays the central role in the description of both sides of the AdS/CFT duality. In section 2.2 one may find a simple pictorial representation of the BDS formula for $F_n^{(1)}$.

Furthermore, in $N = 4$ SYM $F_n^{(1)}$ is given to leading order [29, 30] as a sum of contributions F^{2me} from "2-mass easy" (2me) square diagrams [31], i.e. square diagrams with two external legs at opposite corners on-shell and the other two off-shell, figure 3. Formally,

$$F_n^{(1)} = \sum_{a<b} F^{2me}(\mathbf{p}_a, \mathbf{P}_{ab}, \mathbf{p}_b, \mathbf{P}_{ba}) \tag{1.5}$$

Here $\mathbf{p}_a$ are the n external momenta[1] and $\mathbf{P}_{ab} = \sum_{c=a+1}^{b-1}\mathbf{p}_c$, where we assume that $\mathbf{p}_{a+n} \equiv \mathbf{p}_a$. The two lower case arguments $\mathbf{p}_a$ of F^{2me} are on-shell ($\mathbf{p}_a^2 = 0$), while the other two are in general off-shell.

[1] Throughout the paper, we use the bold font for $4d$ vectors, while arrows are used for $2d$ vectors.

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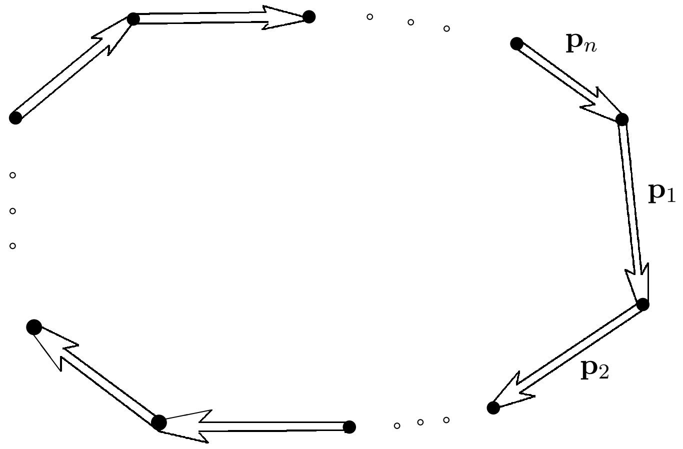


Figure 1: A formal polygon Π formed by the external momenta $\mathbf{p}_a$ (one can call it a Wilson loop in the dual momentum space). It plays a surprisingly important role in the description of both perturbative and strong coupling sides of gauge/string duality.

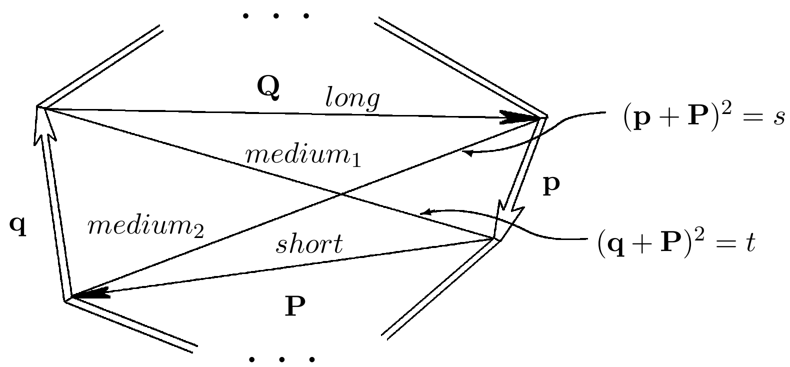


Figure 2: The 4-cluster formed by two non-intersecting edges of the polygon Π, it also contains four vertices (hence the name) and four diagonals. ("Diagonals" are those of Π, two of the four are actually *sides* of the quadrilateral.) From these four diagonals one is "long", another "short" and two "medium" — associated respectively with $\mathbf{P}^2$, $\mathbf{Q}^2$ (or vice versa), s and t. "Long" and "short" refer to the smallest number of edges of Π in between the ends of diagonal. The contribution of the 4 cluster to the dilogarithmic part of the BDS formula (2.2) is $\frac{1}{2}Li_2\left(1 - \frac{\mathbf{P}^2\mathbf{Q}^2}{st}\right)$ where $\log\frac{\mathbf{P}^2\mathbf{Q}^2}{st} = \tau_l + \tau_s - \tau_{m1} - \tau_{m2}$. Dilogarithmic contribution does not distinguish long and short diagonals. The logarithmic one does, see figure 5.

Each $F^{2me}(\mathbf{p},\mathbf{P},\mathbf{q},\mathbf{Q})$ can be expressed through dilogarithmic functions of four invariant scalars $s = (\mathbf{p}+\mathbf{P})^2$, $t = (\mathbf{p}+\mathbf{Q})^2$, $\mathbf{P}^2$, $\mathbf{Q}^2$. A particularly interesting

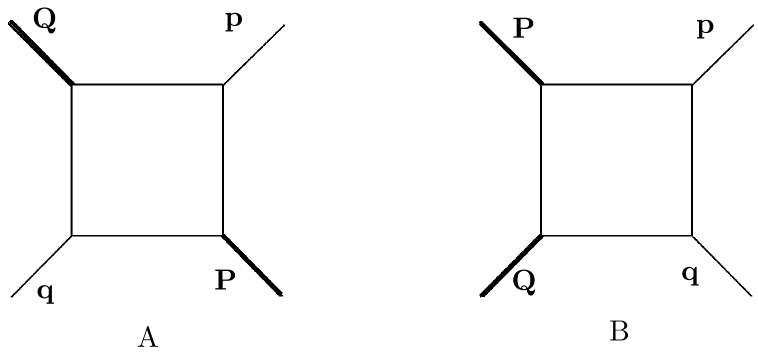


Figure 3: Ordinary box Feynman diagrams for a massless scalar field in $4 + 2\epsilon$ dimensions with four external momenta. Two external momenta are on-shell, two are off-shell: $\mathbf{p}^2 = \mathbf{q}^2 = 0$, $\mathbf{P}^2 \neq 0$, $\mathbf{Q}^2 \neq 0$. In "easy" (or 2-mass easy) box (A) off-shell momenta are at opposite corners, in "heavy" box (B) they are adjacent. Only "easy" boxes contribute to $F_n^{(1)}$ in (1.5).

dilogarithimic representation is [32]:

$$F^{2me}(\mathbf{p},\mathbf{P},\mathbf{q},\mathbf{Q}) \sim \frac{1}{\epsilon^2(-s)^\epsilon} + \frac{1}{\epsilon^2(-t)^\epsilon} + \frac{1}{\epsilon^2(-\mathbf{P}^2)^\epsilon} + \frac{1}{\epsilon^2(-\mathbf{Q}^2)^\epsilon} + \tag{1.6}$$

$$+Li_2(1-as) + Li_2(1-at) - Li_2(1-a\mathbf{P}^2) - Li_2(1-a\mathbf{Q}^2) \tag{1.7}$$

with

$$a = \frac{s + t - \mathbf{P}^2 - \mathbf{Q}^2}{st - \mathbf{P}^2\mathbf{Q}^2}. \tag{1.8}$$

(d) As shown recently in [9], the above sum of dilogarithms is actually a double contour integral (the leading contribution to the Wilson-loop average [22]), only in T-dual coordinates in the target space along a polygon, formed by the external momenta $\mathbf{p}_a$ of the scattering process:

$$F_n^{(1)} = \oint_\Pi \oint_\Pi \frac{dy^\mu dy'_\mu}{(y-y')^{2+\epsilon}} \tag{1.9}$$

Equation (1.9) constitutes a purely geometric formulation of the BDS formula for $F_n^{(1)}$. Together with a Bethe-ansatz description [25] of the function $\gamma(\lambda)$ it should provide a particularly satisfactory solution of planar $N = 4$ SYM.

1.2 String theory side

The situation on the String Theory side of the AdS/CFT correspondence looks at the moment less optimistic. Let us briefly review the current understanding.

(e) According to the AdS/CFT duality [3, 4] the geometric integral (1.9) should in fact coincide with another geometric quantity: an area of a minimal surface in AdS

$$F_n^{(1)} = \text{Minimal area} \tag{1.10}$$

with boundary defined by the external momenta. After a T-duality transformation of the n-point function the boundary conditions become Dirichlet and state that the boundary of the surface is the same polygon Π, figure 1, formed by the external momenta $\mathbf{p}_a$ and located on the boundary of AdS [1].

Classically, the *minimal* area (defined by the Nambu-Goto string action) can be rewritten as the classical action of the AdS σ-model in conformal gauge [1]. The world-sheet σ-model equations of motion are

$$\begin{aligned} \Delta z &= zL, \\ \Delta \mathbf{v} &= \mathbf{v}L \\ z^2 L - (\partial z)^2 &= (z\partial \mathbf{v} - \mathbf{v}\partial z)^2 \end{aligned} \tag{1.11}$$

where $\Delta \equiv \partial^2/\partial u_1^2 + \partial^2/\partial u_2^2 \equiv \partial^2$ is the Laplacian on the world-sheet, described by the coordinates $\vec{u} = (u_1, u_2)$, $-\infty < u_i < +\infty$, while z and $\mathbf{v}$ are coordinates of AdS, related to the T-dual (y^μ, r) and the embedding ones $(\mathbf{Y}, Y_\pm)$ by

$$\mathbf{v} = \frac{\mathbf{y}}{r} = \mathbf{Y}, \quad z = \frac{1}{r} = Y_+, \quad \frac{r^2 - \mathbf{y}^2}{r} = Y_- \tag{1.12}$$

(f) Let us concentrate on the $n = 4$ case, a case in which considerable progress has been made in [1]. As we demonstrate in section 3, in this case there is a whole class of solutions with constant L, some of which are related by SO(4, 2) transforms to the Alday-Maldacena solution [1]. They are

$$z = \sum_{a=1}^{n} z_a e^{\vec{k}_a \vec{u}} \qquad \mathbf{v} = \sum_{a=1}^{n} \mathbf{v}_a e^{\vec{k}_a \vec{u}} \tag{1.13}$$

The four 2-vectors $\vec{k}_a$ all have the same length $\vec{k}_a^2 = L$ and are directed along the diagonals of a rectangular. The four 4-vectors $\mathbf{v}_a$ are related to the external momenta by

$$\mathbf{p}_a = \frac{\mathbf{v}_{a+1}}{z_{a+1}} - \frac{\mathbf{v}_a}{z_a} \tag{1.14}$$

This relation can be considered as defining $\mathbf{v}_a$ for given $\mathbf{p}_a$ and z_a, while the only remaining constraint imposed by the equations of motion on the four parameters z_a is

$$z_1 z_3 s + z_2 z_4 t = 1 \tag{1.15}$$

With constant L, the action (minimal surface area) $\int L d^2 u$ looks independent of external momenta, but actually the integral diverges and requires regularization. According to [1], a special dimensional regularization defines properly the surface area and reproduces the expression of $F_n^{(1)}$ conjectured in [2], but it breaks the AdS-structure of the model, together with its symmetries and integrability, perhaps in an unnecessarily violent manner. See [12] for an extension of [1] to $1/\sqrt{\lambda}$ corrections (at this stage only divergent terms, excluded from $F_n^{(1)}$, are examined) and discussion of potential problems beyond one loop caused by this version of ϵ-regularization.

Nevertheless, in section 4 the regularization prescription of [1] is naively applied to our solutions as well. It leads to a regularized minimal area in the form of an integral, which depends only on z. This means that solutions with different $\{z_a\}$'s, even if they are SO(4, 2) transforms of each other, can give rise to different "areas" after regularization: what may be considered as a new kind of *anomaly*. Then, as usual in anomalous theories, we minimized the resulting expression over the moduli space of solutions under the constraint (1.15). The area of the minimal surface obtained depends, of course, on s and t because of the constraint and remarkably enough, it reproduces exactly the Alday-Maldacena result [1].

A few general remarks and speculations about the present program are in order here.

First, for $n > 4$ the Lagrangian density L for configurations of the form (1.13) is no longer constant, so that the latter may at best be considered an approximate solution — a "trial function" — which might provide an almost but not exactly minimal surface. Exact solutions of the SO(4, 2) sigma-model [34, 35], allowing for growing asymptotics, remain to be found. The vast majority of studies in the field of sigma-models are concentrated on two issues: Lax representation and finite-action (instanton-like) solutions. By contrast, what is needed here are solutions with infinite action to make regularization necessary. Moreover, the target space is non-compact and in the standard σ-model coordinates the solutions of interest are exponentially growing. It is not a big surprise that they have not been studied thoroughly in the literature.

Second, if the regularization scheme has to break the integrability of the sigma model, then the Whitham theory [36] may be relevant, since it is well-known [37] that renormalization group (RG) flows in the vicinity of integrable systems are well described in terms of Whitham hierarchies. At the same time AdS geometry itself should provide a reasonable description of the same RG behavior [38], so that one can probably stay within the pure integrable framework.

Third, whatever regularization is used, the resulting integral will have the general form

$$\int d^2 u L_\epsilon z^\epsilon \tag{1.16}$$

A key step would be to recognize in the finite part of that integral the same kind of *bilinear* structure that exists in its counterpart formula (1.5) or its equivalent *double*-contour integral (1.9).

Finally, one might expect that the hoped for relation

$$\oint_{\partial S} \oint_{\partial S} \frac{dy^\mu dy'_\mu}{(y-y')^{2+\epsilon}} = \text{Area}_\epsilon \text{of a minimal surface } S \tag{1.17}$$

advertized by the AdS/CFT approach, is one between two purely geometric quantities, and should not require any reference to quantum field theory in order to be formulated and proved. However, the *regularized* area on the r.h.s. still needs to be defined in geometric terms. Perhaps, the puzzling relation (1.17) can itself be used as a clue to such a definition.

Also, the "area" on the r.h.s. is not well defined until a regularization prescription is clearly formulated and the resulting "anomaly" problem is resolved.

This concludes our introductory remarks, as well as a brief presentation of our results. The rest of the paper contains detailed explanations of the statements made above and is organized as follows. In section 2 we expand upon the Gauge Theory side and give a potentially useful pictorial representation of the BDS formula for $F_n^{(1)}$. Section 3 contains the presentation of our multi-parameter class of solutions of the AdS σ-model for $n = 4$. In section 4 we use the regularization prescription employed in [1] and compute the minimal area, as a function of the parameters of the solutions. Upon minimization with respect to the moduli we obtain the Alday-Maldacena result. The final section contains a brief summary of our results, a review of open questions related to the present work and suggestions of possible directions for further study.

2. Properties of $F^{(1)}$

2.1 The BDS formula

$F_n^{(1)}$ in (1.4) is a function of invariant variables $t_{ab} = \left(\sum_{c=a}^{b-1} \mathbf{p}_c\right)^2 = (\mathbf{p}_a + \mathbf{P}_{ab})^2$. These t-variables are nothing but squares of diagonals in a polygon Π, figure 4, which is closed due to momentum conservation $\sum_{a=1}^n \mathbf{p}_a = 0$. Given this association of t-variables with diagonals, it is also natural to denote $t_{ab} = t_a^{[b-a]}$ where $[b-a]$ is the *size* of the diagonal: the number of polygon sides that it embraces. Of course, $t_{ab} = t_{ba}$, $t_a^{[2]} = s_{a,a+1}$ and one can restrict diagonal sizes r by $r \leq n/2$. Of all t variables only $3n - 4 - 6 = 3n - 10$ are actually independent (3 stands for three independent components of a null-vector, 4 constraints are imposed by 4-momentum conservation, $\sum_{a=1}^n \mathbf{p}_a = 0$, 10 is the number of Lorentz rotations and translations in 4 dimensions, which all act on p-variables as long as $n \geq 4$), this is, however, not important for our purposes below.

According to [2], in the $N = 4$ supersymmetric gauge theory $F_n^{(1)}$ decomposes into a sum of terms, each depending on only four out of all t-variables. Some of these terms are dilogarithms, while others are squares of ordinary logarithms. The BDS formula (eqs.(4.59)-(4.63) of [2], in a slightly different notation) states that

$$F_n^{(1)} = D_n + L_n^{(1)} + L_n^{(2)} + \text{const} \tag{2.1}$$

where

$$D_n = -\frac{1}{4}\sum_{a=1}^{n}\sum_{r=2}^{n-4} Li_2\left(1 - \frac{t_a^{[r]}t_{a+1}^{[r+2]}}{t_a^{[r+1]}t_{a+1}^{[r+1]}}\right), \tag{2.2}$$

and

$$L_n^{(1)} = -\frac{1}{2}\sum_{a=1}^{n}\sum_{r=2}^{[n/2]-1}\left(\tau_a^{[r+1]} - \tau_a^{[r]}\right)\left(\tau_a^{[r+1]} - \tau_{a+1}^{[r]}\right) \tag{2.3}$$

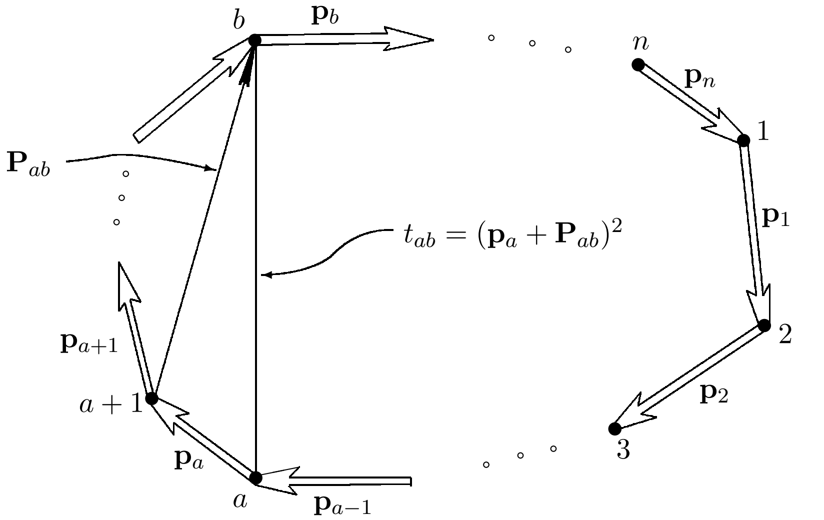


Figure 4: A formal polygon Π made by the external momenta $\mathbf{p}_a$, the same as in figure 1. The picture shows the labeling of vertices, edges and diagonals used throughout the text. Note that diagonals appear only through their squares t_{ab}. All vectors are Minkowskian, and in the $N = 4$ SYM theory, all $\mathbf{p}_a$ are null $\mathbf{p}_a^2 = 0$.

with $\tau_a^{[r]} = \log(-t_{a,a+r})$. The remaining logarithmic term is different for even and odd n, namely

$$\text{for even} \quad n = 2m: \qquad L_n^{(2)} = -\frac{1}{8}\sum_{a=1}^{n}\left(\tau_a^{[m]} - \tau_{a+m+1}^{[m]}\right)\left(\tau_{a+1}^{[m]} - \tau_{a+m}^{[m]}\right) \tag{2.4}$$

while

$$\text{for odd} \quad n = 2m+1: \qquad L_n^{(2)} = -\frac{1}{4}\sum_{a=1}^{n}\left(\tau_a^{[m]} - \tau_{a+m+1}^{[m]}\right)\left(\tau_{a+1}^{[m]} - \tau_{a+m}^{[m]}\right) \tag{2.5}$$

where $m = \big[n/2\big]$. Note that in (2.4) each $\tau_a^{[m]}$ appears twice, as $\tau_a^{[m]}$ and $\tau_{a+m}^{[m]}$, but this is not the case in (2.5) – this explains the difference in the coefficients in front of the sums.

The dilogarithmic part of (2.1) is actually a sum over 4-clusters (figure 2) in a polygon Π, (figure 4) of the quantity

$$\frac{1}{2}\sum_{\mathbf{p},\mathbf{q}} Li_2\left(1 - \frac{\mathbf{P}^2\mathbf{Q}^2}{st}\right) \tag{2.6}$$

A 4-cluster, figure 2, is formed by two non-adjacent edges $\mathbf{p}_a = \mathbf{p}$ and $\mathbf{p}_b = \mathbf{q}$ in Π and consists of the four diagonals of Π, connecting the corresponding four vertices a, $a+1$, $a+r$, $a+r+1$. Squared lengths of these four diagonals are $t_{a,b+1} = t_a^{[r+2]} = \mathbf{Q}^2$, $t_{ab} = t_a^{[r+1]} = s$, $t_{a+1,b+1} = t_{a+1}^{[r+1]} = t$ and $t_{a+1,b} = t_{a+1}^{[r]} = \mathbf{P}^2$, where $\mathbf{P} = \mathbf{P}_{ab}$ and $\mathbf{Q} = \mathbf{P}_{ba}$ and $r = j-i-1$ is the size of the shortest diagonal in the cluster. It is indeed a diagonal only for $r \geq 2$ and such 4-clusters — and thus dilogarithmic contributions to $F_n^{(1)}$ — exist for $n \geq 6$.

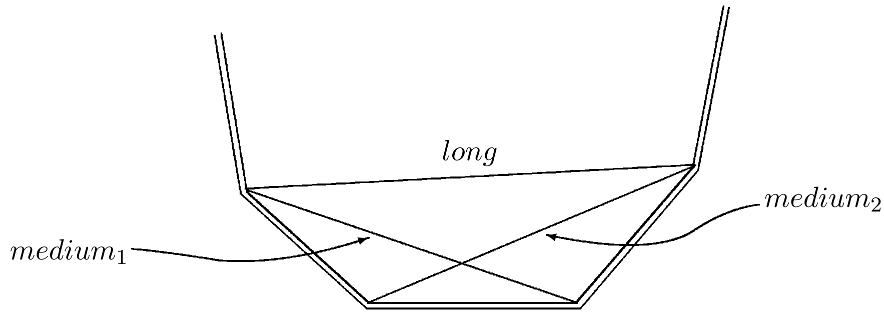


Figure 5: Degenerate 4-cluster: one of the would-be diagonals is actually an edge of Π, thus it is null and the corresponding t would be zero. Degenerate clusters do not contribute to the dilogarithmic part of the BDS formula (2.2), but do contribute to the logarithmic part (2.3), see figure 2.

Note that, from the point of view of scattering theory, eq. (2.1) is better than (2.6) because the summation over different appearances of each t_{ab} is already performed. However, the older formula (2.6) better suites *our* purposes, since it describes more nicely the internal structures of the problem.

2.2 A pictorial representation

The simple structure of the somewhat sophisticated formula (2.1) can be revealed in pictures. As already mentioned, eq. (2.1) represents $F_n^{(1)}$ as a sum over all 4-clusters of the polygon, with the single exception of peculiar 5-clusters, figure 10, contributing for odd n. *Degenerate* 4-clusters where the smallest diagonal coincides with an edge, figure 5, do not contribute to the dilogarithmic piece (2.2), but only to the logarithmic one (2.3). Since for $n = 4$ and $n = 5$ all 4-clusters are degenerate, there are no dilogarithms at all in $F_4^{(1)}$ and $F_5^{(1)}$.

The dilogarithm is made out of the four diagonals of a non-degenerate 4-cluster. The four τ-parameters associated with the four diagonals are summed with the signs shown in figure 6, the sum is exponentiated to give a ratio of t's, which is further subtracted from unity and used as an argument of a dilogarithm.

Logarithms are a little bit trickier. The smallest diagonal does not contribute (thus degenerate clusters are allowed), while the largest one contributes twice. The contribution of a 4-cluster is the product of the two differences of τ for the largest diagonal and τ's for the two medium diagonals, figure 7.

Additional complications arise when the largest diagonal in the cluster is the main diagonal of the polygon Π. The situation is somewhat different for n even and odd.

For even n the main diagonal is just the diameter of Π. When the largest diagonal of the cluster is a diameter, it appears in two different 4-clusters — to the right and to the left of the diameter, figure 8, — and both clusters should be included into the sum. Finally,

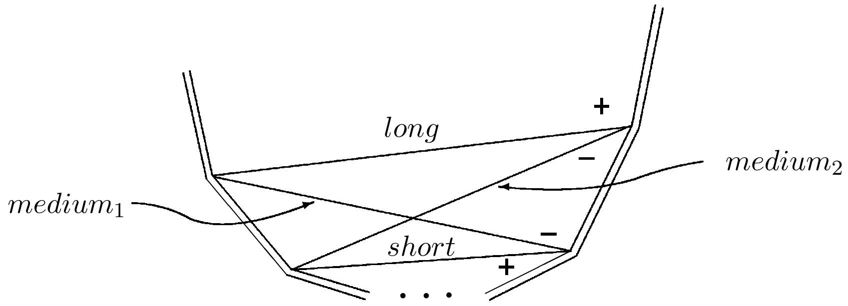


Figure 6: Relative signs of the contributions of different diagonals of a given cluster to an argument of dilogarithm in (2.2): $\log\frac{\mathbf{P}^2\mathbf{Q}^2}{st} = \tau_l + \tau_s - \tau_{m1} - \tau_{m2}$, see figure 2.

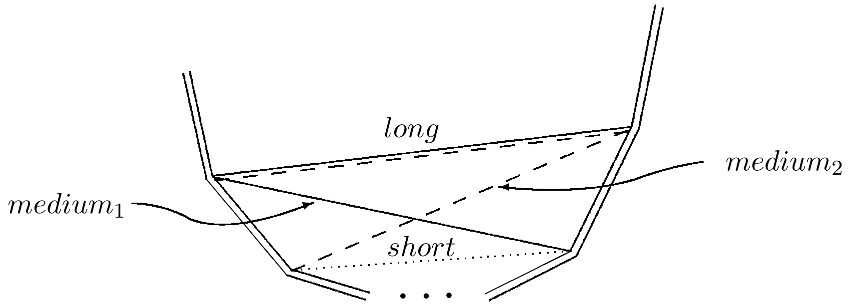


Figure 7: Generic 4-cluster with well defined "long" diagonal. Its contribution to the logarithmic part of the BDS formula (2.3) is $\frac{1}{2}(\tau_l - \tau_{m_1})(\tau_l - \tau_{m_2})$ and does not depend on the "short" diagonal. Therefore this contribution is well defined even for the degenerate clusters in figure 5. *Generic* 4-cluster is fully defined by its long diagonal, so that each long diagonal contributes once to the logarithmic part of the BDS formula and twice — to its dilogarithmic part. Exceptions from this rule are the *longest* (main) diagonals of Π, see figures 8- 10.

some clusters do not have the largest diagonal: they contain adjacent *main* diagonals (diameter for even n), figure 9. The contribution of such a cluster is just the square of differences between τ's for the two main diagonals, the smaller diagonals do not contribute (they could coincide with edges if such a cluster happens to be degenerate – which is indeed the case for $n = 4$ and $n = 5$).

For odd n contributing (along with figure 7) are peculiar 5-clusters with four *main* diagonals, see figure 10.

2.3 Asymptotics for nearly light-like diagonals

If some $t_b^{[r]}$ is much smaller than all other t's, then the arguments of the corresponding dilogarithms are large, so that the dilogarithms become squares of logarithms and cancel against the logarithmic terms. Indeed, let us take some $t_b^{[r]} \to 0$ — this means that the corresponding $\tau_b^{[r]} \to -\infty$ and see what happens to (2.1). If it appears in the numerator of the argument of a dilogarithm, nothing happens because $Li_2(z)$ is regular near $z = 1$.

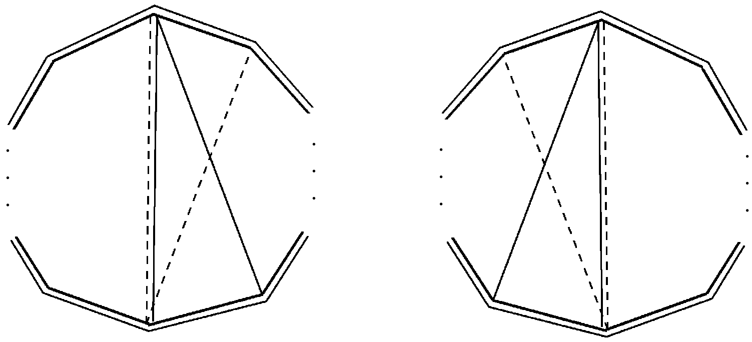


Figure 8: Special 4-clusters in which the long diagonal is the *longest* (main) in Π: diameter of Π for n even. In this picture the cluster has a single longest diagonal. Then its contribution to the logarithmic part of the BDS formula (2.3) is the same as in figure 7, but there are *two* clusters associated with this longest diagonal – one to its right and the other to its left.

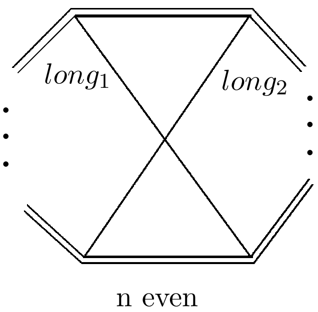
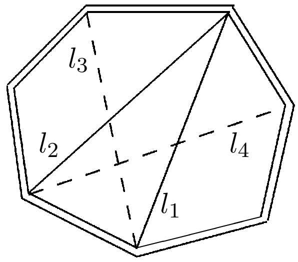


Figure 9: Special 4-clusters where the long diagonals are the *longest* (main) in Π. The case of even n is shown, when *two* long diagonals are diameters. The contribution of this 4-cluster to the BDS formula (2.3)+(2.4) is $\frac{1}{4}\Big(\tau_{l_1} - \tau_{l_2}\Big)^2$ — very different from the one in figure 7.

However, if it appears in the denominator, the dilogarithm blows up and gets expressed through ordinary logarithms via $Li_2(z) \sim -\frac{1}{2}(\log z)^2$ for $|z| \to \infty$. Taking this into account, we can write (note that our $t_b^{[r]}$ appears in dilogarithms twice: also as $t_{b+r}^{[n-r]}$, hence the extra factor of two in the dilogarithmic contributions in the first line of (2.7) below.)

$$\begin{aligned}
&\frac{2}{8}\Big(\tau_b^{[r-1]} + \tau_{b-1}^{[r+1]} - \underline{\tau_b^{[r]}} - \tau_{b-1}^{[r]}\Big)^2 + \frac{2}{8}\Big(\tau_{b+1}^{[r-1]} + \tau_b^{[r+1]} - \tau_{b+1}^{[r]} - \underline{\tau_b^{[r]}}\Big)^2 - \\
&-\frac{1}{2}\Big(\tau_b^{[r+1]} - \underline{\tau_b^{[r]}}\Big)\Big(\tau_b^{[r+1]} - \tau_{b+1}^{[r]}\Big) - \frac{1}{2}\Big(\underline{\tau_b^{[r]}} - \tau_b^{[r-1]}\Big)\Big(\underline{\tau_b^{[r]}} - \tau_{b+1}^{[r-1]}\Big) - \\
&-\frac{1}{2}\Big(\tau_{b-1}^{[r+1]} - \tau_{b-1}^{[r]}\Big)\Big(\tau_{b-1}^{[r+1]} - \underline{\tau_b^{[r]}}\Big)
\end{aligned} \tag{2.7}$$

It is easy to see that all terms with the underlined quantity cancel. As a result, this $t_b^{[r]}$ completely drops out from $F_n^{(1)}$ for $r \geq 3$. For $r = 2$ one should make a separate calculation (since we ignored the restriction $r > 2$ in (2.7) and kept terms with $r - 1$), and it is easy



n odd

Figure 10: In the case of odd n the longest (main) diagonals enter through a 5-cluster (involving five vertices of Π). The contribution to the BDS formula (2.5) is made from four longest diagonals and is given by $\frac{1}{4}\Big(\tau_{l_1} - \tau_{l_2})(\tau_{l_3} - \tau_{l_4}\Big)$ — again very different from the one in figure 7. This is the only case where 5-clusters contribute to the BDS formula.

to see that, in contrast to $t_b^{[r]}$ with $r > 2$, the small $t_b^{[2]}$ leads to a singularity in $F_n^{(1)}$ of the form, figure 11:

$$F_n^{(1)} = -\frac{1}{2}\underline{\tau_b^{[2]}}\Big(\tau_{b-1}^{[2]} + \tau_{b+1}^{[2]} - \tau_{b-1}^{[3]} - \tau_b^{[3]}\Big) + \text{ terms finite as } \tau_b^{[r]} \to -\infty \tag{2.8}$$

eq. (2.8) is not directly applicable also when $t_b^{[r]} \to 0$ with maximal $r = \big[n/2\big]$, such $t^{[r]}$s enter (2.1) in a more sophisticated way; moreover, they are different for even and odd n, see figures 9- 10. Still, it is easy to demonstrate that no singularity in $F_n^{(1)}$ occurs when $t^{[r]} \to 0$ with $r = \big[n/2\big]$ — unless $\big[n/2\big] = 2$, i.e. $n = 4$ or $n = 5$. Note that at $n = 4$ eq. (2.8) also requires a correction:

$$F_4^{(1)} = \frac{1}{4}\Big(\log\frac{s}{t}\Big)^2 = \frac{1}{4}(\log s)^2 - \frac{1}{2}\log s\,\log t \; + \text{ terms finite as } \log s \to -\infty \tag{2.9}$$

coming from figure 9.

To analyse the singularities at large t one has to take into account all relations among different t's and is beyond the scope of this paper.

2.4 Boxes: two dilogarithmic representations

We now turn to the analysis of individual box contributions to $F_n^{(1)}$. Denoting $\mathbf{p}^2 = \mathbf{q}^2 = 0$, $s = (\mathbf{p} + \mathbf{P})^2$, $t = (\mathbf{q} + \mathbf{P})^2$ for a particular box, associated with the 4-cluster in figure 2, we obtain for the easy-box Feynman diagram, figure 3.A, with a massless scalar field in the loop:

$$F^{2me}(s, t; \mathbf{P}^2, \mathbf{Q}^2) \sim \int_0^1 \frac{d\beta_1 d\beta_2 d\beta_3 d\beta_4 \delta(1 - \beta_1 - \beta_2 - \beta_3 - \beta_4)}{(-s\beta_1\beta_3 - \mathbf{P}^2\beta_3\beta_4 - t\beta_2\beta_4 - \mathbf{Q}^2\beta_1\beta_2)^{2+\epsilon}} \tag{2.10}$$

The calculation of this integral is somewhat tedious. As a result, there are two essentially different formulas for this quantity: the first one [31] is convenient to make contact with (2.6), the second one [32] with (2.16), which will be studied in the next section 2.5 and, furthermore, with the contour integral (1.9). It is actually here, at (2.10), that the

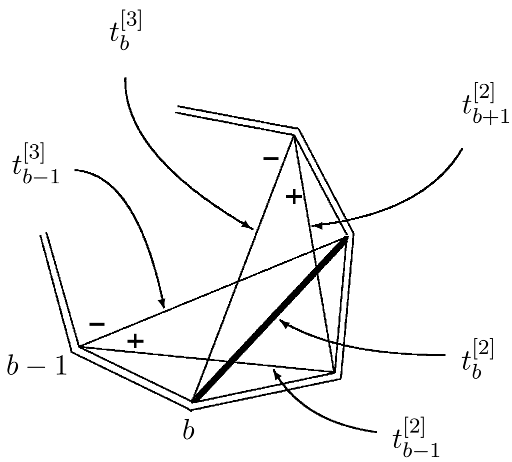


Figure 11: The single 4-cluster contributing to the asymptotic (2.8) of $F_n^{(1)}$ as $t_b^{[2]} \longrightarrow 0$ (the corresponding diagonal is shown by thick line and becomes light-like in the limit). This is the only type of singularities that $F_n^{(1)}$ has at small $t_a^{[r]}$. Other clusters can also have singularities, but they cancel between dilogarithmic and logarithmic contributions to (2.1). $t_a^{[r]}$ with $r = 2$ are distinguished because they never appear in denominators in arguments of dilogarithms.

roads split: one leads to the BDS formula in the form (2.1) and the other to its geometric representation (1.9).

The BDK formula (eq. (4.44) of [31]) states that

$$F^{2me}(s,t;\mathbf{P}^2,\mathbf{Q}^2) = \frac{2i}{4\pi^2}\frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{st-\mathbf{P}^2\mathbf{Q}^2}\left\{\frac{1}{\epsilon^2}\left[\frac{1}{(-s)^\epsilon}+\frac{1}{(-t)^\epsilon}-\frac{1}{(-\mathbf{P}^2)^\epsilon}-\frac{1}{(-\mathbf{Q}^2)^\epsilon}\right]+\right.$$
$$+Li_2\left(1-\frac{\mathbf{P}^2\mathbf{Q}^2}{st}\right)-Li_2\left(1-\frac{\mathbf{P}^2}{s}\right)-Li_2\left(1-\frac{\mathbf{P}^2}{t}\right)$$
$$\left.-Li_2\left(1-\frac{\mathbf{Q}^2}{s}\right)-Li_2\left(1-\frac{\mathbf{Q}^2}{t}\right)\right\} \tag{2.11}$$

The first dilogarithm is exactly the same as in (2.6), other dilogarithms cancel non-trivially between different 4-clusters in the sum (1.5). For $n = 4$ when both $\mathbf{P}^2 = \mathbf{Q}^2 = 0$ and $n = 5$ when either $\mathbf{P}^2 = 0$ or $\mathbf{Q}^2 = 0$ the first term disappears and there are no dilogarithms in the answer at all (for $n = 5$ this still involves non-trivial cancelation of other dilogarithms between different 4-clusters).

The DN formula (eq. (71) of [32]) states that

$$F^{2me}(s,t;\mathbf{P}^2,\mathbf{Q}^2) = \frac{2i}{4\pi^2}\frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{st-\mathbf{P}^2\mathbf{Q}^2}\cdot$$
$$\cdot\left\{\frac{1}{\epsilon^2}\left[\left(\frac{-s-i\varepsilon}{4\pi\mu^2}\right)^{-\epsilon}+\left(\frac{-t-i\varepsilon}{4\pi\mu^2}\right)^{-\epsilon}-\left(\frac{-\mathbf{P}^2-i\varepsilon}{4\pi\mu^2}\right)^{-\epsilon}-\left(\frac{-\mathbf{Q}^2-i\varepsilon}{4\pi\mu^2}\right)^{-\epsilon}\right]+\right.$$
$$\left.+Li_2\big(1-a(s+i\varepsilon)\big)+Li_2\big(1-a(t+i\varepsilon)\big)-Li_2\big(1-a(\mathbf{P}^2+i\varepsilon)\big)-Li_2\big(1-a(\mathbf{Q}^2+i\varepsilon)\big)\right\} \tag{2.12}$$

with

$$a = \frac{s + t - \mathbf{P}^2 - \mathbf{Q}^2}{st - \mathbf{P}^2\mathbf{Q}^2}$$

The equivalence of (2.11) and (2.12) shown in [32], is based on basic dilogarithmic identities such as

$$\begin{aligned} Li_2(z) &= \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \quad Li_2(0) = 0, \quad Li_2(1) = \zeta_2 = \frac{\pi^2}{6}, \\ Li_2(z) + Li_2(1-z) &= -\log(1-z)\log z - \frac{\pi^2}{6}, \\ Li_2(z) + Li_2(1/z) &= -\frac{1}{2}\big(\log(-z)\big)^2 - \frac{\pi^2}{6} \end{aligned} \tag{2.13}$$

The rather lengthy proof is presented in appendix A of [32]. The equivalence of (2.11) and (2.1) is one of the subjects of [2]. We skip these derivations here.

2.5 Double integral [22] along a polygon [9]

As shown in [9], the sum (1.5) of the easy-box diagrams, if represented in the form (2.12), is nothing but the double contour integral (1.9) along the auxiliary polygon Π:

$$F_n^{(1)} = \ (1.5) \ \stackrel{(2.12)}{=} \oint_{\Pi} \oint_{\Pi} \frac{dy^{\mu} dy'_{\mu}}{(\mathbf{y} - \mathbf{y}')^{2+\epsilon}} \tag{2.14}$$

Indeed, (1.9) is a sum of contributions coming from pairs of segments in Π. There are three different types of pairs, which we briefly consider following [9]. In this section τ_p and τ_q parameterize the segments $\mathbf{p}$ and $\mathbf{q}$.

2.5.1 One null-segment p

No contribution because $dydy' = \mathbf{p}^2 d\tau d\tau'$ in the numerator vanishes for $\mathbf{p}^2 = 0$.

2.5.2 Two adjacent null-segments p and q

$$\int_0^1 \int_0^1 \frac{(\mathbf{pq}) d\tau_p d\tau_q}{(\tau_p \mathbf{p} + \tau_q \mathbf{q})^{2+\epsilon}} = \frac{(\mathbf{pq})^{-\epsilon/2}}{2^{1+\epsilon/2}} \int_0^1 \int_0^1 \frac{d\tau_p d\tau_q}{(\tau_p \tau_q)^{1+\epsilon/2}} = \frac{2}{\epsilon^2} (2\mathbf{pq})^{-\epsilon/2}, \ \epsilon < 0 \tag{2.15}$$

2.5.3 Two non-adjacent null-segments p and q

$2\mathbf{pq} = u = \mathbf{P}^2 + \mathbf{Q}^2 - s - t$

$$\begin{aligned} &\int_0^1 \int_0^1 \frac{(\mathbf{pq}) d\tau_p d\tau_q}{\Big(\tau_p \mathbf{p} + \mathbf{P} + (1-\tau_q)\mathbf{q}\Big)^{2+\epsilon}} \stackrel{\tau_q \to 1-\tau_q}{=} \\ &= \frac{1}{2} \int_0^1 \int_0^1 \frac{(\mathbf{P}^2 + \mathbf{Q}^2 - s - t)\ d\tau_p d\tau_q}{\Big(\mathbf{P}^2 + (s - \mathbf{P}^2)\tau_p + (t - \mathbf{P}^2)\tau_q + (\mathbf{P}^2 + \mathbf{Q}^2 - s - t)\tau_p\tau_q\Big)^{1+\frac{\epsilon}{2}}} = \\ &= \frac{1}{2} \int_0^1 \frac{(\mathbf{P}^2 + \mathbf{Q}^2 - s - t)\ d\tau_p}{(\mathbf{P}^2 + \mathbf{Q}^2 - s - t)\tau_p + t - \mathbf{P}^2} \log \frac{t + (\mathbf{Q}^2 - t)\tau_p}{\mathbf{P}^2 + (s - \mathbf{P}^2)\tau_p} + O(\epsilon) = \\ &= Li_2(1 - as) + Li_2(1 - at) - Li_2(1 - a\mathbf{P}^2) - Li_2(1 - a\mathbf{Q}^2) + O(\epsilon) \end{aligned} \tag{2.16}$$

with

$$a = \frac{\mathbf{P}^2 + \mathbf{Q}^2 - s - t}{\mathbf{P}^2\mathbf{Q}^2 - st} \tag{2.17}$$

Since this contribution is finite, we do not preserve its ϵ-dependence.

3. Minimal surfaces with non-planar polygon boundaries

In contrast to the situation in AdS spaces, the study of minimal surfaces in flat space is an old branch of mathematics with close links to the theory of Riemann surfaces and many impressive results. The first non-trivial minimal surfaces, helicoid and catenoid, were found by Meusnier in 1776 [39]. Among the next pioneers in the field were Scherk (1834) and Schwarz (1890). In particular, Schwarz solved the problem of finding the minimal bounding surface of a skew quadrilateral. For a survey of these old results and a presentation of the state-of-the-art in the theory of minimal surfaces in Euclidean space see [40, 41].

In string theory the issue of minimal surfaces was first raised by D. Gross and P. Mende in [42] in an application to scattering amplitudes at high energies: the saddle-point approximation to the Veneziano and Koba-Nielsen formulas for scattering amplitudes is associated with minimal surfaces in Euclidean (Minkowskian) space of appropriate dimension. They studied the case $n = 4$ with the vectors $\mathbf{p}_a$ lying in a $2+1$-dimensional subspace R^3_{++-}.

3.1 Generalities

3.1.1 Ambiguities in the formulation of the problem:

The problem of finding a minimal area, although it sounds as being a well-formulated problem, it contains a few ambiguities due to the necessity of regularizing the infinite area that emerges. While *before* regularization all the formulations are equivalent, this is not necessarily the case *after* the regularization is performed.

The various formulations of this problem use:

- **Different actions:** Polyakov/Green-Schwarz (σ-model) and Nambu-Goto. The minimal area formulation is in terms of the Nambu-Goto action, while motivation for the AdS geometry (with the fifth coordinate in the target-space associated with the Liouville field in the first-quantized formalism) is more transparent in the σ-model formulation. According to [33], the Nambu-Goto action with appropriate boundary terms should be used.

- **Different regularizations:** shift and dimensional. Shift from the AdS boundary is better conceptually, it is compatible with the RG interpretation of AdS, it preserves supersymmetry and integrability; an unusual version of dimensional regularization is used in [1]. For any regularization, the problem is that solutions of regularized equations are much more difficult to find and one needs to get the most from solutions of the *non-regularized* ones: in this respect the approach of [1] was successful, but, as often happens in such circumstances, it is hard to generalize.

- **Different descriptions of the target-space:** original (x^μ, z) coordinates and T-dual coordinates (y^μ, r) can be naturally used, the metric in both cases has the Poincare form, $(dz^2 + dx^\mu dx_\mu)/z^2$ and $(dr^2 + dy^\mu dy_\mu)/r^2 = (dr^2 - dy_0^2 + dy_1^2 + dy_2^2 + dy_3^2)/r^2$, but boundary conditions are imposed at $z \to \infty$ and $r \to 0$ and they are Dirichlet conditions in the $(r, \mathbf{y})$ case. According to [1], transition to dual variables also eliminates boundary terms, which had to be added in the Nambu-Goto action [33].

- **Different coordinates at the boundary:** say, $y_\pm = y_0 \pm y_1, y_2, y_3$ or y_0, y_1, y_2, y_3 or y_0, ξ_1, ξ_2, y_3 etc. This is a purely technical issue, but many papers differ mostly in these choices.

Literature: Most considerations in the literature are devoted to a single cusp (see, however, [33]) formed by two null-lines [33, 18, 27, 7]. In [18] this was done in rapidity coordinates, and actually four cusps were implicitly involved. In [1] this fact was exploited to construct a solution with rhombic projection of the boundary on the (y_1, y_2) plane.

3.1.2 The Nambu-Goto action with $y_3 = 0$ in y_1, y_2 projection:

In this case it is convenient to choose the gauge with the two world-sheet coordinates identified with the coordinates (y_1, y_2). Then, the Nambu-Goto action is

$$S_{NG} = \int\int \frac{dy_1 dy_2}{r^2}\sqrt{H}, \tag{3.1}$$

$$H = 1 - (\partial_1 y)^2 - (\partial_2 y)^2 + (\partial_1 r)^2\left[1 - (\partial_2 y)^2\right] + (\partial_2 r)^2\left[1 - (\partial_1 y)^2\right] + 2\partial_1 r \partial_2 r \partial_1 y \partial_2 y \tag{3.2}$$

The equations of motion are

$$\partial_i \frac{\partial_i y}{r^2\sqrt{H}} + \partial_2 \frac{\partial_1 r(\partial_1 r \partial_2 y - \partial_2 r \partial_1 y)}{r^2\sqrt{H}} - \partial_1 \frac{\partial_2 r(\partial_1 r \partial_2 y - \partial_2 r \partial_1 y)}{r^2\sqrt{H}} = 0 \tag{3.3}$$

$$\partial_i \frac{\partial_i r}{r^2\sqrt{H}} + \partial_2 \frac{\partial_1 y(\partial_1 r \partial_2 y - \partial_2 r \partial_1 y)}{r^2\sqrt{H}} - \partial_1 \frac{\partial_2 \mathbf{y}(\partial_1 r \partial_2 y - \partial_2 r \partial_1 y)}{r^2\sqrt{H}} + \frac{2}{r^2\sqrt{H}} = 0 \tag{3.4}$$

When approaching the boundary given by the segment perpendicular to a vector $\vec{q}$, $\vec{q}\vec{y} = q_1 y_1 + q_2 y_2 = 1$, the coordinate r behaves as $r \sim \sqrt{\vec{q}\vec{y} - 1}$. Poles in $\partial_\perp r$ are canceled by zeroes of $[1 - (\partial_{||} y)^2]$. These zeroes arise if boundary segments are null-vectors.

3.1.3 Nambu-Goto vs σ-model action

Formally at the classical level the two formulations should be equivalent, but with a non-trivial $2d$ metric. In order to put the $2d$ metric into the conformal gauge, one needs to make a general coordinate transformation.

It is worth noting, though, that only for the σ-model action one can perform a T-duality transformation $(r, \mathbf{x}) \to (z, \mathbf{y})$ with $z = 1/r$ and $\partial_i \mathbf{x} = z^2 \epsilon_{ij} \partial_j \mathbf{y}$. Indeed, this transformation does not preserve the shape of the 2×2 tensor (induced metric)

$$G_{ij}(r, \mathbf{x}) = \frac{\partial_i r \partial_j r + \partial_i \mathbf{x} \partial_j \mathbf{x}}{r^2} \tag{3.5}$$

Moreover, it changes the determinant $\det G_{ij}(r, \mathbf{x}) \neq \det G_{ij}(z, \mathbf{y})$, thus, the Nambu-Goto action is formally not T-invariant. What is invariant, is the trace: $\delta^{ij} G_{ij}(r, \mathbf{x}) = \delta^{ij} G_{ij}(z, \mathbf{y})$, and this is enough to guarantee T-invariance of the σ-model action and of the associated Polyakov and Green-Schwarz actions.

3.1.4 Equations of motion for the AdS σ-model

The equations of motion for the σ-model action

$$S_\sigma = \int L d^2 u, \qquad L = \frac{(\partial r)^2 + (\partial \mathbf{y})^2}{r^2} \tag{3.6}$$

(in appropriate coordinates (u_1, u_2) on the world sheet) are:

$$\begin{aligned} \partial \left(\frac{\partial r}{r^2} \right) &= -\frac{L}{r} \\ \partial \left(\frac{\partial \mathbf{y}}{r^2} \right) &= 0 \end{aligned} \tag{3.7}$$

and in coordinates $z = 1/r$, $\mathbf{v} = \mathbf{y}/r$ they acquire the form

$$\begin{aligned} \Delta z &= zL, \\ \Delta \mathbf{v} &= \mathbf{v} L \\ z^2 L - (\partial z)^2 &= (z \partial \mathbf{v} - \mathbf{v} \partial z)^2 \end{aligned} \tag{3.8}$$

For $L = const$ solutions of the first two equations are sums of exponentials

$$\begin{aligned} z &= \sum_{a=1}^{n} z_a e^{\vec{k}_a \vec{u}}, \\ \mathbf{v} &= \sum_{a=1}^{n} \mathbf{v}_a e^{\vec{k}_a \vec{u}} \end{aligned} \tag{3.9}$$

with $\vec{k}_a^2 = L$.

3.1.5 Boundary conditions

Since all vectors $\vec{k}_a$ have equal lengths, the boundary conditions can also be easily satisfied (see figure 12): when $(\vec{k}_b + \vec{k}_{b+1})\vec{u} \longrightarrow \infty$ only two terms with $a = b - 1$ and $a = b$ contribute, and the dependence on the orthogonal variable $t_b = e^{(\vec{k}_{b+1} - \vec{k}_b)\vec{u}/2}$ gets simple:

$$\mathbf{y} = \frac{\mathbf{v}_{b+1} t_b + \mathbf{v}_b t_b^{-1}}{z_{b+1} t_b + z_b t_b^{-1}} \tag{3.10}$$

and implies linear relations between components of the $4d$ vector $\mathbf{y}$:

$$(z_{b+1} v_b^\nu - z_b v_{b+1}^\nu) y^\mu - (z_{b+1} v_b^\mu - z_b v_{b+1}^\mu) y^\nu = (v_{b+1}^\mu v_b^\nu - v_b^\mu v_{b+1}^\nu) \tag{3.11}$$

As t_b varies from 0 to ∞, the vector $\mathbf{y}$ changes from $\frac{\mathbf{v}_b}{z_b}$ to $\frac{\mathbf{v}_{b+1}}{z_{b+1}}$, and the boundary conditions imply that this change is exactly the b-th external momentum $\mathbf{p}_b$:

$$\Delta_b \mathbf{y} = \frac{\mathbf{v}_{b+1}}{z_{b+1}} - \frac{\mathbf{v}_b}{z_b} = \mathbf{p}_b \tag{3.12}$$

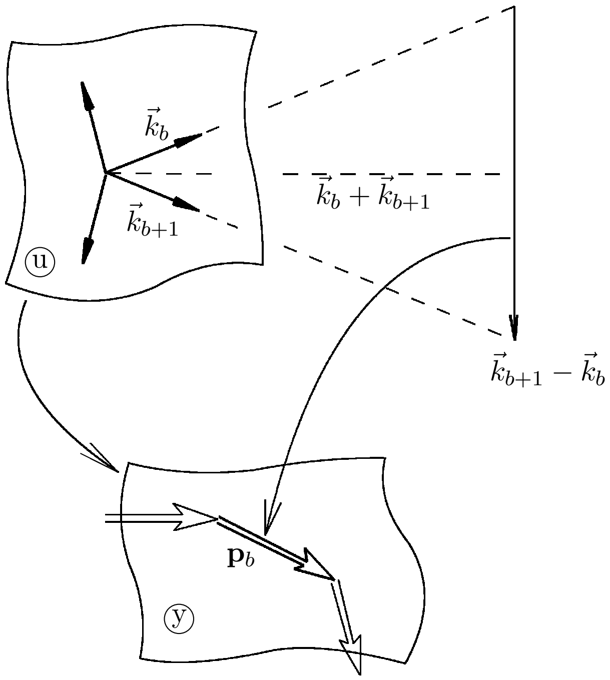


Figure 12: Vectors $\vec{k}_a$ lie in the Euclidean $2d$ u-plane. They all have the same length $\vec{k}_a^2 = L = 2$. Far at infinity along the bisector $\vec{k}_b + \vec{k}_{b+1}$ and orthogonal to it is a line, directed along $\vec{k}_{b+1} - \vec{k}_b$. This is mapped by the fields z and $\mathbf{v}$ onto a vector $\mathbf{p}_b$ — an edge of the polygon Π on the boundary ($z = \infty$) of the AdS target space.

Equivalently, the boundary conditions are:

$$z_b \mathbf{v}_{b+1} - z_{b+1}\mathbf{v}_b = z_b z_{b+1}\mathbf{p}_b \tag{3.13}$$

We remind that, in our notation, $b + n \equiv b$.

3.1.6 The third equation: a problem

While the first two equations in (3.8) are easily satisfied by the ansatz (3.9) with $L = const$, this is not generally true for the third equation. Indeed, after substitution of (3.9) it becomes

$$\sum_{a,b} z_a z_b \Big(L - (\vec{k}_a\vec{k}_b)\Big) E_{a+b} = \sum_{\substack{a<b\\ c<d}} (\mathcal{P}_{ab}\mathcal{P}_{cd})(\vec{k}_{ab}\vec{k}_{cd}) E_{a+b+c+d} \tag{3.14}$$

where $E_{a_1+\ldots+a_m} = e^{(\vec{k}_{a_1}+\ldots+\vec{k}_{a_m})\vec{u}}$, $\vec{k}_{ab} = \vec{k}_a - \vec{k}_b$ and

$$\mathcal{P}_{ab} = z_a\mathbf{v}_b - z_b\mathbf{v}_a = z_a z_b\big(\mathbf{p}_a + \mathbf{p}_{a+1} + \ldots + \mathbf{p}_{b-1}\big) = z_a z_b\big(\mathbf{p}_a + \mathbf{P}_{ab}\big) \tag{3.15}$$

Note that $\vec{k}_a$ and $\vec{k}_{ab}$ are $2d$ vectors on the world sheet, while $\mathbf{p}_a$ and $\mathcal{P}_{ab}$ are $4d$ vectors in the (T-dualized) target space. Incidentally, $\mathbf{P}_{ab}$ are the vectors which enter eq. (2.10).

The problem with eq. (3.14) is that the set of exponentials $E_{a+b+c+d}$ on the r.h.s. is larger then that of E_{a+b} on the l.h.s. The first term that causes trouble is $(a,b,c,d) = (a,a,a+1,a-1)$: there is no associated $E = e^{(\vec{k}_{a+1}+2\vec{k}_a+\vec{k}_{a-1})\vec{u}}$ on the l.h.s., while on the r.h.s. it appears with the coefficient

$$z_{a+1}z_a^2 z_{a-1}\Big[(\vec{k}_a - \vec{k}_{a-1})(\vec{k}_a - \vec{k}_{a+1})\Big](2\mathbf{p}_a\mathbf{p}_{a+1}) \tag{3.16}$$

The last bracket is nothing but $t_{a,a+2} = (\mathbf{p}_a + \mathbf{p}_{a+1})^2 \neq 0$, while the square bracket with the scalar product of $2d$ vectors vanishes for all a only for $n = 4$ and with the $\vec{k}$-vectors pointing along the diagonals of a rectangle.

Beyond $n = 4$ the problem is somewhat reminiscent of Serre relations in group theory, only here it does not seem to have a simple solution. So, we shall concentrate on the $n = 4$ case.

3.2 The example of quadrilaterals, $n = 4$

3.2.1 An anomaly

$4d$ Lorentz invariance allows one to convert the original geometry, associated with the four momenta $\mathbf{p}_a$, to any convenient form with just two independent parameters s and t. To begin with, it allows to choose y-coordinates so that $y_3 = 0$ and consider projections of the momenta on the (y_1, y_2) plane, where they form an ordinary quadrilateral. In order to provide a closed line in the y_0-projection the side lengths of the edges of this quadrilateral should satisfy an additional constraint

$$l_1 \pm l_2 \pm l_3 \pm l_4 = 0. \tag{3.17}$$

A non-planar quadrilateral in y-space arises if the signs are taken to be $(+-+-)$. Thus for further calculations one can choose any 2-parametric family of quadrilaterals in (y_1, y_2) plane with $l_1 + l_3 = l_2 + l_4$. For the two independent parameters $\sqrt{s}$ and $\sqrt{t}$ one can take the two diagonals (in space-time, not in the projection on the (y_1, y_2) plane — this is the same only when $l_1 = l_2$, i.e. for the cases of rhombus and kite).

In [1], the shape was chosen to be rhombic, $l_1 = l_2 = l_3 = l_4$, but we do not impose such restriction in this section. This provides an additional self-consistency check: the result (regularized minimal area) should be Lorentz invariant and independent of the particular shape of the quadrilateral. This is not fully guaranteed, because any particular solutions of the equations of motion spontaneously break Lorentz invariance and — while obviously restored in the non-regularized problem (indeed, $L = 2$ for all solutions) — it can still remain broken after regularization.

Unfortunately this is what eventually happens, as we shall see below: this self-consistency check actually fails. The reason will be that the *regularized* minimal area, defined according to the recipe of [1], depends on the lifting from the boundary conditions ($\{\mathbf{p_a}\}$ or, equivalently, $\{\mathbf{v_a}\}$) to the space of solutions (parameterized by ($\{z_a\}$) and there is no obvious canonical choice of this lifting, i.e. it does not produce a unique answer. The Lorentz invariance itself can be easily restored if one asks the lifting to remain intact under Lorentz rotation, but there is still no distinguished way to obtain an answer for given s and t. As

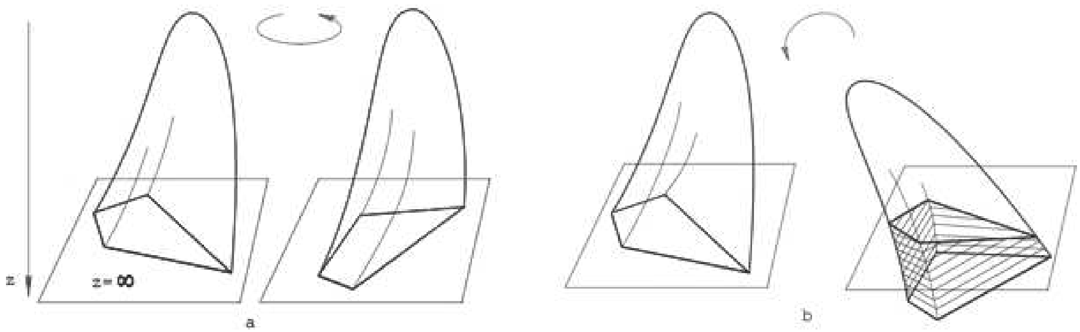


Figure 13: Minimal surfaces, connected (a) by a Lorentz transformation from SO(3,1) and (b) by some non-Lorentzian transformation from SO(4,2). From the point of view of a remote observer, located at finite z, boundary conditions with three different polygons: P_1, its Lorentz rotated version P_2 and an essentially different P_3 should be considered equivalent. However, as is clear from the picture, the actual areas should not coincide: they differ by the area of the shadowed domain, which may not be negligible. This picture also shows that the anomaly, considered in this paper, may not be an artifact of the ϵ-regularization of [1]: it can be present for conventional $r^2 \to r^2 + \epsilon^2$ regularization as well.

shown in [1], a lifting exists that reproduces the BDS formula, but it is unclear what are the *a priori* reasons to choose this particular lifting and what are the ways to generalize it to other (non-rhombic) shapes — other than just an SO(4,2) rotation of the same *ad hoc* prescription. See figure 13 for a pictorial description of this anomaly.

The resolution of the anomaly is well known. When the result is a non-trivial function on moduli space, one should integrate over it or, classically, find its extremum. This is exactly what we shall do in section 4.6.

3.2.2 Equation and solutions

The boundary conditions (3.13) are

$$\begin{aligned}
\mathcal{P}_{12} &= z_1\mathbf{v}_2 - z_2\mathbf{v}_1 = z_1 z_2 \mathbf{p}_1, \\
\mathcal{P}_{23} &= z_2\mathbf{v}_3 - z_3\mathbf{v}_2 = z_2 z_3 \mathbf{p}_2, \\
\mathcal{P}_{34} &= z_3\mathbf{v}_4 - z_4\mathbf{v}_3 = z_3 z_4 \mathbf{p}_3, \\
\mathcal{P}_{41} &= z_4\mathbf{v}_1 - z_1\mathbf{v}_4 = z_4 z_1 \mathbf{p}_4
\end{aligned} \tag{3.18}$$

or

$$\mathbf{p}_a = \frac{\mathbf{v}_{a+1}}{z_{a+1}} - \frac{\mathbf{v}_a}{z_a} \tag{3.19}$$

It follows that

$$\begin{aligned}
\mathcal{P}_{13} = z_1\mathbf{v}_3 - z_3\mathbf{v}_1 = z_1 z_3(\mathbf{p}_1 + \mathbf{p}_2) \\
\mathcal{P}_{24} = z_2\mathbf{v}_4 - z_4\mathbf{v}_2 = z_2 z_4(\mathbf{p}_1 + \mathbf{p}_3)
\end{aligned} \tag{3.20}$$

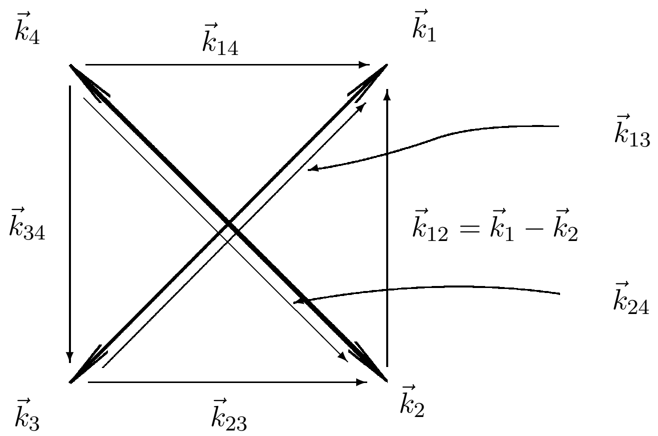


Figure 14: Concrete choice (3.23) of the $\vec{k}$-vectors on the $2d$ u-plane.

As shown in [1], in the case of $n = 4$ L can be constant, as will be verified by the explicit solution below and, following [1], we adjust the scale of $\vec{u}$ so that $L = 2$. The non-trivial (third) equation of motion in (3.8) is

$$2z^2 - (\partial z)^2 = (z\partial\mathbf{v} - \mathbf{v}\partial z)^2 \tag{3.21}$$

In the form (3.14) this equation reads

$$\sum_{a,b=1}^{4} z_a z_b \Big(2 - (\vec{k}_a \vec{k}_b)\Big) E_{a+b} = \sum_{\substack{a<b\\ c<d}} (\mathcal{P}_{ab}\mathcal{P}_{cd})(\vec{k}_{ab}\vec{k}_{cd}) E_{a+b+c+d} \tag{3.22}$$

Let us begin with a special — Z_4-symmetric — choice of the $n = 4$ $2d$ vectors $\vec{k}$ with $\vec{k}^2 = L = 2$, shown in figure 14

$$\vec{k}_1 = (+1,+1), \quad \vec{k}_2 = (+1,-1), \quad \vec{k}_3 = (-1,-1) = -\vec{k}_1, \quad \vec{k}_4 = (-1,+1) = -\vec{k}_2, \tag{3.23}$$

Clearly, in this case one can relabel indices 3 and 4 of the exponentials E_a to -1 and -2 respectively, what we shall do in some of the formulas. It is also natural to label $E_{1+4} = E_{1-2}$ and $E_{1-1} = E_0 = 1$. In this notation

$$\begin{aligned} z\partial\mathbf{v} - \mathbf{v}\partial z = \sum_{a,b} \vec{k}_{ab} \otimes \mathcal{P}_{ab} E_{a+b} = z_1 z_2 \vec{k}_{12} \otimes \mathbf{p}_1\, E_{1+2} + z_1 z_3 \vec{k}_{13} \otimes (\mathbf{p}_1 + \mathbf{p}_2)\, E_0 + \\ - z_1 z_4 \vec{k}_{14} \otimes \mathbf{p}_4\, E_{1-2} + z_2 z_3 \vec{k}_{23} \otimes \mathbf{p}_2\, E_{-1+2} + \\ + z_2 z_4 \vec{k}_{24} \otimes (\mathbf{p}_2 + \mathbf{p}_3)\, E_0 + z_3 z_4 \vec{k}_{34} \otimes \mathbf{p}_3\, E_{-1-2} \end{aligned} \tag{3.24}$$

and the equation becomes

$$
\begin{aligned}
&8\big(z_1z_3+z_2z_4\big)E_0+4\Big(z_1z_2E_{1+2}+z_2z_3E_{-1+2}+z_3z_4E_{-1-2}+z_1z_4E_{1-2}\Big)=\\
&\quad=\Big((z_1z_3)^2\vec{k}_{13}^2(\mathbf{p}_1+\mathbf{p}_2)^2+(z_2z_4)^2\vec{k}_{24}^2(\mathbf{p}_2+\mathbf{p}_3)^2+\\
&\qquad+2z_1z_2z_3z_4\big[(\vec{k}_{12}\vec{k}_{34})(\mathbf{p}_1\mathbf{p}_3)-(\vec{k}_{14}\vec{k}_{23})(\mathbf{p}_2\mathbf{p}_4)\big]\Big)E_0+\\
&\qquad+2z_1z_2\Big(z_1z_3(\vec{k}_{12}\vec{k}_{13})\big[\mathbf{p}_1(\mathbf{p}_1+\mathbf{p}_2)\big]+z_2z_4(\vec{k}_{12}\vec{k}_{24})\big[\mathbf{p}_1(\mathbf{p}_2+\mathbf{p}_3)\big]\Big)E_{1+2}+\\
&\qquad+2z_2z_3\Big(z_1z_3(\vec{k}_{23}\vec{k}_{13})\big[\mathbf{p}_2(\mathbf{p}_1+\mathbf{p}_2)\big]+z_2z_4(\vec{k}_{23}\vec{k}_{24})\big[\mathbf{p}_2(\mathbf{p}_2+\mathbf{p}_3)\big]\Big)E_{-1+2}+\\
&\qquad+2z_3z_4\Big(z_1z_3(\vec{k}_{34}\vec{k}_{13})\big[\mathbf{p}_3(\mathbf{p}_1+\mathbf{p}_2)\big]+z_2z_4(\vec{k}_{34}\vec{k}_{24})\big[\mathbf{p}_3(\mathbf{p}_2+\mathbf{p}_3)\big]\Big)E_{-1-2}-\\
&\qquad-2z_1z_4\Big(z_1z_3(\vec{k}_{14}\vec{k}_{13})\big[\mathbf{p}_4(\mathbf{p}_1+\mathbf{p}_2)\big]+z_2z_4(\vec{k}_{14}\vec{k}_{24})\big[\mathbf{p}_4(\mathbf{p}_2+\mathbf{p}_3)\big]\Big)E_{1-2}
\end{aligned}
\tag{3.25}
$$

Many terms are actually absent from the r.h.s. due to $\mathbf{p}_a^2=0$ or $\vec{k}_a^2=2$ or $\vec{k}_a\vec{k}_{a+1}=0$. Most important, these conditions are enough to exclude terms like $E_{1+1+2+2}$ or E_{1+1} from the r.h.s., which would have no counterparts at all on the l.h.s. More explicitly,

- terms like $E_{1+1+2+2}$ in the quartic part (i.e. at the r.h.s.) do not appear because $\mathbf{p}_a^2=0$,
- terms like $E_{1+1}=E_{(1+2)+(-1+2)}$ in the quartic part (on the r.h.s.) do not appear because of $\vec{k}_a\vec{k}_{a+1}=0$,
- terms like $E_{1+1}=E_{(1+2)+(-1+2)}$ in the quadratic part (on the l.h.s.) do not appear because of $\vec{k}_a^2=2$.

As a result of these cancelations we are left with five equations: those for $E_{\pm1\pm2}$ and $E_0=1$

$$
\begin{array}{llll}
E_0: & z_1z_3+z_2z_4= & (z_1z_3)^2s+(z_2z_4)^2t-z_1z_2z_3z_4u & =(z_1z_3+z_2z_4)\big(z_1z_3s+z_2z_4t\big)\\
E_{1+2}: & z_1z_2= & z_1z_2\big(z_1z_3s-z_2z_4(s+u)\big) & =z_1z_2\big(z_1z_3s+z_2z_4t\big)\\
E_{-1+2}: & z_2z_3= & & z_2z_3\big(z_1z_3s+z_2z_4t\big)\\
E_{-1-2}: & z_3z_4= & z_3z_4\big(-z_1z_3(t+u)+z_2z_4t\big) & =z_3z_4\big(z_1z_3s+z_2z_4t\big)\\
E_{1-2}: & z_1z_4= & -z_1z_4\big(z_1z_3(t+u)+z_2z_4(u+s)\big) & =z_1z_4\big(z_1z_3s+z_2z_4t\big)
\end{array}
\tag{3.26}
$$

Here $s=(\mathbf{p}_1+\mathbf{p}_2)^2=2\mathbf{p}_1\mathbf{p}_2$, $\quad t=(\mathbf{p}_2+\mathbf{p}_3)^2=2\mathbf{p}_2\mathbf{p}_3$, $\quad u=(\mathbf{p}_1+\mathbf{p}_3)^2=2\mathbf{p}_1\mathbf{p}_3=2\mathbf{p}_2\mathbf{p}_4=-s-t$.

All five equations above coincide and are equivalent to the single relation

$$
z_1z_3s+z_2z_4t=1\ ,\quad\text{or}\quad z_1z_3t_{13}+z_2z_4t_{24}=1
\tag{3.27}
$$

3.3 SO$(4,2)$ symmetry between different choices of z_a

Eq. (3.27) defines the common scale of all factors z_a, but does not fix relations between them. Since all the solutions with different choices of $\{z_a\}$ have the same Lagrangian value

$L = 2$, it is clear that they are related by a symmetry. This symmetry is nothing but the conformal group SO(4, 2). Indeed, our z and $\mathbf{v}$ variables are nothing but flat coordinates in $\mathbb{R}^6_{++++--}$,

$$\mathbf{v} = \frac{\mathbf{y}}{r} = \frac{\mathbf{Y}}{R}, \quad z = \frac{1}{r} = \frac{Y_{-1} + Y_4}{R^2} = Y_+, \quad \frac{r^2 - \mathbf{y}^2}{r} = Y_{-1} - Y_4 = Y_-, \tag{3.28}$$

where SO(4, 2) acts linearly and AdS_5 is embedded as a quadratic

$$\mathbf{Y}^2 + Y_+ Y_- = Y_{-1}^2 + Y_0^2 - Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 = R^2 \tag{3.29}$$

The flat metric in $\mathbb{R}^6$ induces the AdS metric in the Poincare form

$$d\mathbf{Y}^2 + dY_+ dY_- = \frac{dr^2 + d\mathbf{y}^2}{r^2} \tag{3.30}$$

A priori, the ansatz (3.9) does not seem to imply anything nice for the $\vec{u}$-dependence of the sixth σ-model coordinate Y_-. However, eq. (3.27) is exactly the condition that Y_- is also a sum of $n = 4$ exponentials,

$$Y_- = \sum_{a=1}^{n} w_a E_a \tag{3.31}$$

Indeed, substituting this expression together with (3.9) into (3.29), one obtains

$$\begin{aligned} 0 &= (z_1 E_1 + z_2 E_2 + z_3 E_3 + z_4 E_4)(w_1 E_1 + w_2 E_2 + w_3 E_3 + w_4 E_4) \\ &\quad -(\mathbf{v}_1 E_1 + \mathbf{v}_2 E_2 + \mathbf{v}_3 E_3 + \mathbf{v}_4 E_4)^2 - 1 \\ &= \sum_{a=1}^{4} \left\{ \left(z_a w_a - \mathbf{v}_a^2 \right) E_a^2 + \left(z_a w_{a+1} + z_{a+1} w_a - 2\mathbf{v}_a \mathbf{v}_{a+1} \right) E_a E_{a+1} \right\} \\ &\quad + \left(w_1 z_3 + w_2 z_4 + w_3 z_1 + w_4 z_2 - 2\mathbf{v}_1 \mathbf{v}_3 - 2\mathbf{v}_2 \mathbf{v}_4 - 1 \right) \end{aligned}$$

The vanishing condition for the first term defines

$$w_a = \frac{\mathbf{v}_a^2}{z_a} \tag{3.32}$$

The coefficient in the second term is then equal to

$$\frac{z_a}{z_{a+1}} \mathbf{v}_{a+1}^2 + \frac{z_{a+1}}{z_a} \mathbf{v}_a^2 - 2\mathbf{v}_a \mathbf{v}_{a+1} = \frac{\left(z_a \mathbf{v}_{a+1} - z_{a+1} \mathbf{v}_a \right)^2}{z_a z_{a+1}} \overset{(3.12)}{=} z_a z_{a+1} {\mathbf{p}_a}^2 = 0$$

and vanishes automatically. Finally, vanishing the last term is exactly the relation (3.27).

Note that the Lorentz transformations SO(3, 1) can change $\mathbf{p}_a$ and the shape of the boundary in target space, but they are not enough to relate solutions with different $\{z_a\}$. The problem is that conformal symmetry and thus the equivalence between different choices of $\{z_a\}$ is broken by dimensional regularization *a la* [1]. We return to a discussion of this *anomaly* problem in section 4 and especially in Subsection 4.6. The resolution of the anomaly problem will lead to the existence of preferable choices for $\{z_a\}$ for given boundary conditions.

3.4 Solutions for $n = 4$

In this subsection we write down explicitly various solutions for the $n = 4$ case in the rapidity coordinates $\xi_1 = \tanh u_1$, $\xi_2 = \tanh u_2$.

3.4.1 The Alday-Maldacena solution [1]

The principal choice of [1] is

$$z_3 = z_1, \quad z_4 = z_2 \tag{3.33}$$

Then equation (3.27) becomes

$$z_1^2 s + z_2^2 t = 1 \tag{3.34}$$

but it still leaves a one-parameter freedom in the choice of z_1 and z_2. Alday and Maldacena fix it by putting

$$z_1^2 = \frac{1}{2s}, \quad z_2^2 = \frac{1}{2t} \tag{3.35}$$

and find it convenient to reexpress everything in intermediate formulas through auxiliary parameters a and b:

$$s = \frac{A^2}{2(1-b)^2}, \quad t = \frac{A^2}{2(1+b)^2}, \quad z_1 = \frac{1-b}{A}, \quad z_2 = \frac{1+b}{A}, \quad A = \frac{a}{2\pi} \tag{3.36}$$

This choice of z_a does not actually restrict the possibility to impose arbitrary boundary conditions and we shall see in a moment that the generic set of external momenta $\{\mathbf{p}_a\}$ can be described with (3.36). However, in [1] the particular choice is made

$$\begin{aligned} r &= a\frac{\sqrt{(1-\xi_1^2)(1-\xi_2^2)}}{1+b\xi_2\xi_2}, \quad & y_0 &= a\frac{\sqrt{1+b^2}\xi_1\xi_2}{1+b\xi_1\xi_2}, \quad y_1 = a\frac{\xi_1}{1+b\xi_1\xi_2}, \\ y_2 &= a\frac{\xi_2}{1+b\xi_1\xi_2}, & y_3 &= 0, \end{aligned} \tag{3.37}$$

where $\xi_1 = \tanh u_1$, $\xi_2 = \tanh u_2$ are the rapidity variables. In these coordinates, the four boundaries of our quadrilateral are at $\xi_{1,2} = \pm 1$, and the whole solution is a mapping of a square in the rapidity plane into the target space. In terms of $y_{1,2}$, the boundaries are at $y_1 + by_2 = \pm a$ and $by_1 + y_2 = \pm a$, i.e. form a *rhombus* with diagonals $\sqrt{s}$ and $\sqrt{t}$. Note that, since $l_1 = l_2 = l_3 = l_4$, the squares of diagonals in space-time (which are actually s and t) coincide with the squares of their projections onto the (y_1, y_2) plane.

¿From (3.37) we can deduce the Nambu-Goto action for this solution: $H = \frac{1+b\xi_1\xi_2}{1-b\xi_1\xi_2}$, $dy_1 \wedge dy_2 = \frac{1-b\xi_1\xi_2}{(1+b\xi_1\xi_2)^3}\, d\xi_1 \wedge d\xi_2$, the equations of motion are again true, and

$$S_{NG} = \int_0^1 \int_0^1 \frac{d\xi_1 d\xi_2}{(1-\xi_1^2)(1-\xi_2^2)} \tag{3.38}$$

obviously coincides with the σ-model action.

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3.4.2 Beyond [1]

Thus, the Alday-Maldacena solution is fixed by three choices: (3.33), (3.35) and (3.37). Actually all three are ambigous and can be deformed, giving rise to new solutions: not a big surprise given the SO(4, 2) symmetry of the problem. One could safely take any family of solutions sufficient to describe arbitrary values of s and t and ignore all the rest – if SO(4, 2) symmetry was not violated by ϵ-regularization. Since only regularized area makes sense, one should actually analyze *all* solutions and see what happens — and the result is unpleasant: the answer depends on the choice. The answer matches the BDS formula for the Alday-Maldacena choice, but they do not match in general. *A priori* no way to distinguish the Alday-Maldacena choice among all others remain unclear. We shall propose a way after regularization in section 4.6.

Needless to say, it is unclear what should be a counterpart of the Alday-Maldacena choice for $n > 4$, where a variety of methods, including approximate and numerical, could be used if one knew *what kind* of solution one should concentrate on. It is to demonstrate all these issues that we proceed with the detailed presentation of other solutions as well.

3.4.3 All z_a equal

Before we proceed with the discussion of the Alday-Maldacena solution and its generalizations to arbitrary quadrilaterals in the target space, we now analyze a much simpler option: when all four z_a are the same, and according to (3.27)

$$z_a = \frac{1}{\sqrt{s+t}} \tag{3.39}$$

Then (3.9) implies that $z = 4\cosh u_1 \cosh u_2$ and in rapidity coordinates,

$$\begin{aligned} r &= a\sqrt{(1-\xi_1^2)(1-\xi_2^2)}, \\ \mathbf{y} &= \alpha + \beta\xi_1 + \gamma\xi_2 + \delta\xi_1\xi_2 \end{aligned} \tag{3.40}$$

where $a = \sqrt{s+t}$ and the four $4d$ vectors

$$\begin{aligned} \alpha &= \frac{a}{4}\Big(\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} + \mathbf{v_4}\Big), \\ \beta &= \frac{a}{4}\Big(\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3} - \mathbf{v_4}\Big), \\ \gamma &= \frac{a}{4}\Big(\mathbf{v_1} - \mathbf{v_2} - \mathbf{v_3} + \mathbf{v_4}\Big), \\ \delta &= \frac{a}{4}\Big(\mathbf{v_1} - \mathbf{v_2} + \mathbf{v_3} - \mathbf{v_4}\Big) \end{aligned} \tag{3.41}$$

At the four boundaries of the square $\xi_1, \xi_2 = \pm 1$, $\mathbf{y}$ form four segments of straight lines, which should coincide with the four external momenta: these are our familiar boundary conditions (3.12),

$$\mathbf{v}_{a+1} - \mathbf{v}_a = \frac{\mathbf{p}_a}{\sqrt{s+t}} \tag{3.42}$$

Putting, say, $\xi_1 = 1$ one obtains $\mathbf{y} = (\alpha+\beta) + (\gamma+\delta)\xi_2$, which varies along the segment $\xi_2 = [-1, 1]$ from $(\alpha+\beta) - (\gamma+\delta)$ to $(\alpha+\beta) + (\gamma+\delta)$, i.e.

$$\Delta_1 \mathbf{y} = 2(\gamma+\delta) = -\mathbf{p_1} \tag{3.43}$$

Similarly

$$\begin{aligned}\Delta_2 \mathbf{y} &= 2(\beta+\delta) = \mathbf{p_4},\\ \Delta_{-1} \mathbf{y} &= 2(\gamma-\delta) = \mathbf{p_3},\\ \Delta_{-2} \mathbf{y} &= 2(\beta-\delta) = -\mathbf{p_2}\end{aligned} \tag{3.44}$$

along the boundaries $\xi_2 = 1$, $\xi_1 = -1$ and $\xi_2 = -1$ respectively, so that equivalently

$$\begin{aligned}\beta &= -\frac{1}{4}(\mathbf{p}_2 - \mathbf{p}_4),\\ \gamma &= -\frac{1}{4}(\mathbf{p}_1 - \mathbf{p}_3),\\ \delta &= -\frac{1}{4}(\mathbf{p}_1 + \mathbf{p}_3) = \frac{1}{4}(\mathbf{p}_2 + \mathbf{p}_4)\end{aligned} \tag{3.45}$$

while α is a total shift of $\mathbf{y}$ and remains unspecified by the boundary conditions — like the weighted common shift of all $\mathbf{v}$-vectors, $\mathbf{v}_a \to \mathbf{v}_a + z_a \mathbf{w}$.

Thus, it is explicitly shown that a solution exists with all z_a for arbitrary choice of external momenta $\mathbf{p}_a$. With the same choice of momenta as in [1],

$$\mathbf{p}_1 = \frac{2}{1-b^2}\left(\sqrt{1+b^2},\ 1,\ -b,\ 0\right) \tag{3.46}$$

$$\mathbf{p}_2 = \frac{2}{1-b^2}\left(-\sqrt{1+b^2},\ -b,\ 1,\ 0\right) \tag{3.47}$$

$$\mathbf{p}_3 = \frac{2}{1-b^2}\left(\sqrt{1+b^2},\ -1,\ b,\ 0\right) \tag{3.48}$$

$$\mathbf{p}_4 = \frac{2}{1-b^2}\left(-\sqrt{1+b^2},\ b,\ -1,\ 0\right) \tag{3.49}$$

one obtains the solution with

$$\mathbf{v}_1 = -\frac{1}{4}\left(1,\ \frac{1}{\sqrt{1+b^2}},\ \frac{1}{\sqrt{1+b^2}},\ 0\right) \tag{3.50}$$

$$\mathbf{v}_2 = -\frac{1}{4}\left(-1,\ -\frac{1}{\sqrt{1+b^2}},\ \frac{1+2b}{\sqrt{1+b^2}},\ 0\right) \tag{3.51}$$

$$\mathbf{v}_3 = -\frac{1}{4}\left(1,\ \frac{2b-1}{\sqrt{1+b^2}},\ \frac{2b-1}{\sqrt{1+b^2}},\ 0\right) \tag{3.52}$$

$$\mathbf{v}_4 = -\frac{1}{4}\left(-1,\ \frac{1+2b}{\sqrt{1+b^2}},\ -\frac{1}{\sqrt{1+b^2}},\ 0\right) \tag{3.53}$$

This gives

$$y_0 = a\xi_1\xi_2, \qquad y_1 = \frac{a}{\sqrt{1+b^2}}(b + b\xi_1 - \xi_2), \qquad y_2 = \frac{a}{\sqrt{1+b^2}}(-b + \xi_1 + b\xi_2) \tag{3.54}$$

Along with these expressions describing the square (at $b = 0$) and rhombus choices [1] (see figures 15.A,B), one can consider a kite, figure 15.C, or arbitrary asymmetric skew quadrilateral satisfying $l_1 + l_3 = l_2 + l_4$, figure 15.D.

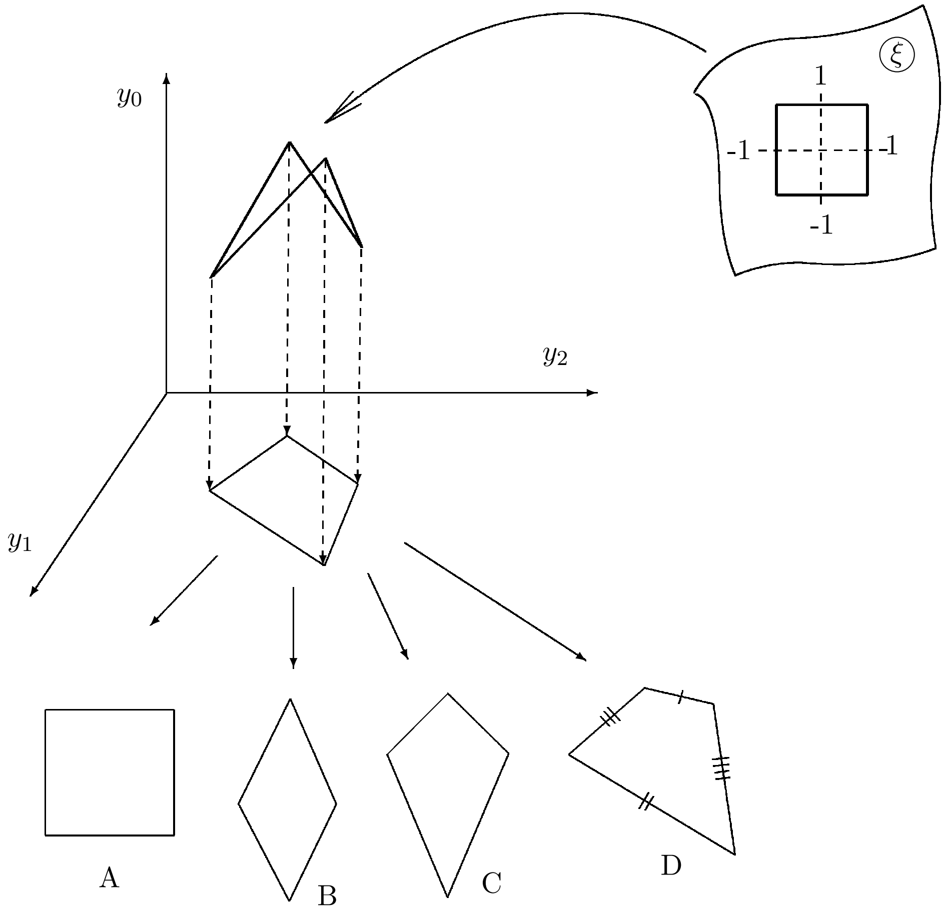

Figure 15: Examples of quadrilaterals on the boundary of the AdS target space at $r = 0$ and also with $y_3 = 0$, together with their projections onto the (y_1, y_2) plane: A. Square. B. Rhombus. C. Kite. D. generic skew quadrilateral.

3.4.4 The Alday-Maldacena solution revisited

Coming back to the Alday-Maldacena solution, it is now clear, that *as for any other choice of* $\{z_a\}$ it could describe an *arbitrary* configuration of $\mathbf{p}_a$, not only rhombic. Indeed, if we impose (3.33) and (3.36), we can still write instead of (3.37)

$$r = a\sqrt{(1-\xi_1^2)(1-\xi_2^2)},$$
$$\mathbf{y} = \frac{\alpha + \beta\xi_1 + \gamma\xi_2 + \delta\xi_1\xi_2}{1 + b\xi_1\xi_2} \tag{3.55}$$

At the same boundaries $\xi_{1,2} = \pm 1$ we now have straight segments in $\mathbf{y}$ space, parameterized in a slightly more complicated way. For example at $\xi_1 = 1$

$$\mathbf{y} = \frac{(\alpha+\beta) + (\gamma+\delta)\xi_2}{1 + b\xi_2} \tag{3.56}$$

It is indeed a straight line, an intersection of three $3d$ hyperplanes in $4d$ space given by $\mathbf{q}\mathbf{y} = c$ with $\Big((\gamma+\delta) - b(\alpha+\beta)\Big)\mathbf{q} = 0$ and $c = \mathbf{q}(\alpha+\beta)$. The corresponding vector

$$\Delta_1\mathbf{y} = \frac{2(\gamma+\delta)}{1-b^2} - \frac{2b(\alpha+\beta)}{1-b^2} = -\mathbf{p}_1 \tag{3.57}$$

Similarly, the analogues of (3.44) and (3.45) are:

$$\begin{aligned}
\Delta_2\mathbf{y} &= \frac{2(\beta+\delta)}{1-b^2} - \frac{2b(\alpha+\gamma)}{1-b^2} = \mathbf{p_4}, \\
\Delta_{-1}\mathbf{y} &= \frac{2(\gamma-\delta)}{1-b^2} + \frac{2b(\alpha-\beta)}{1-b^2} = \mathbf{p_3}, \\
\Delta_{-2}\mathbf{y} &= \frac{2(\beta-\delta)}{1-b^2} + \frac{2b(\alpha-\gamma)}{1-b^2} = -\mathbf{p_2}
\end{aligned} \tag{3.58}$$

and

$$\begin{aligned}
\beta &= -\frac{1-b^2}{4}(\mathbf{p}_2 - \mathbf{p}_4 + b(\mathbf{p}_1 - \mathbf{p}_3)), \\
\gamma &= -\frac{1-b^2}{4}(\mathbf{p}_1 - \mathbf{p}_3 + b(\mathbf{p}_2 - \mathbf{p}_4)), \\
b\alpha - \delta &= -\frac{1-b^2}{4}(\mathbf{p}_2 + \mathbf{p}_4) = \frac{1-b^2}{4}(\mathbf{p}_1 + \mathbf{p}_3)
\end{aligned} \tag{3.59}$$

with α and δ not uniquely specified.

The Alday-Maldacena original choice (3.49) being substituted into (3.59) with $b \neq 0$, reproduces (3.37), while substitution of the same (3.49) into (3.45) gives a different-looking but equivalent solution — before ϵ-regularization!

3.4.5 An example of one-cusp solution

The next examples concern one-cusp solutions (in fact, it is rather *a corner* than *a cusp*, the name seems to be due to historical reasons).

Choice 3: Direction towards a cusp is along one of the vectors $\vec{k}$, say, $\vec{k}_1$. In this limit E_b dominates over all other exponentials. However, one can make use of the freedom to shift $\mathbf{v}_a$'s without changing the boundary conditions to put $\mathbf{v}_1 = 0$ (this can not be done for all corners/cusps of the polygon at once, but is allowed in the case of a single isolated cusp). Then the two adjacent exponentials should be kept in the formulas for $\mathbf{v}$ and the **single-cusp** solution (3.9) becomes

$$\begin{aligned}
z &= z_1E_1 + O(E_{\pm 2}), \\
\mathbf{v} &= \mathbf{v}_2E_2 + \mathbf{v}_4E_{-2} + O(E_{-1})
\end{aligned} \tag{3.60}$$

Since $\mathbf{v}_1 = 0$, the boundary conditions are now $z_1\mathbf{v}_2 = z_1z_2\mathbf{p}_1$ and $-z_1\mathbf{v}_4 = z_1z_4\mathbf{p}_4$. Now using eq. (3.27), one obtains $1 - z_1z_3s = z_2z_4t = (2\mathbf{p}_1\mathbf{p}_4)z_2z_4 = -2\mathbf{v}_2\mathbf{v}_4/z_1^2$. In the literature, the remaining freedom is typically used to fix $z_3 = 0$, which leads to

$$2\mathbf{v}_2\mathbf{v}_4 = z_1^2 \tag{3.61}$$

The solution (3.60) still can be chosen in different ways: depending on the choice of $\mathbf{v}_2$ and $\mathbf{v}_4$, which are restricted by a single constraint (3.61). In particular, in [18, 33, 27, 1]

$$\begin{aligned} z_1 &= \sqrt{2}, & v_2^+ &= 1, & v_4^- &= 1, \\ z &= \sqrt{2}E_1 = \sqrt{2}e^{u_1+u_2}, & v^+ &= v^0 + v^1 = E_2 = e^{u_1-u_2}, \\ v^- &= v^0 - v^1 = E_{-2} = e^{-u_1+u_2}, & v^2 &= v^3 = 0, \end{aligned} \tag{3.62}$$

and in [7]

$$\begin{aligned} z_1 &= \sqrt{2}, & v_2^+ &= 1, & v_4^- &= 1, & v_2^- &= \gamma^2, & v_2^2 &= \gamma \\ z &= \sqrt{2}E_1, & v^+ &= E_2, & v^- &= E_{-2} + \gamma^2 E_2, & v^2 &= \gamma E_2, & v^3 &= 0, \end{aligned} \tag{3.63}$$

so that in coordinates

$$(+,-,2,3) \qquad \mathbf{p}_1 = \frac{\mathbf{v}_2}{z_2} = \frac{(1,\gamma^2,\gamma,0)}{z_2}, \quad \mathbf{p}_4 = \frac{\mathbf{v}_4}{z_4} = \frac{(0,1,0,0)}{z_4} \tag{3.64}$$

One can certainly make many other choices (all equivalent in the above sense, as long as we stay in AdS_5).

It is instructive to look once again at the equations of motion for the solution (3.62): $z = \sqrt{2}E_1 = \sqrt{2}e^{u_1+u_2}$, $y^\pm = v^\pm/z = E_{-1\pm 2}$, i.e. $y^+ = e^{-2u_2}$, $y^- = e^{-2u_1}$ and $r = z^{-1} = \sqrt{2y^+y^-} = \sqrt{2}e^{-u_1-u_2}$. From the two terms on the l.h.s. of the equation

$$\partial_1(z^2\partial_1 y^\pm) + \partial_2(z^2\partial_2 y^\pm) = 0 \tag{3.65}$$

only one survives for each component of y, because, say, $\partial_2 y^- = 0$. Further, $\partial_1 y^- = -2y^-$, then multiplication by z^2 (division by r^2) converts y^- into $1/y^+$, which is finally annihilated by the action of the second ∂_1.

3.5 More quadrilateral solutions: another kind of deformation and a hidden symmetry

Since the only essential property of quadrilaterals that allowed (3.9) to be an exact solution was $(\vec{k}_a - \vec{k}_{a-1})(\vec{k}_a - \vec{k}_{a+1}) = 0$, it is clear that a further generalization is possible: the vectors $\vec{k}_{1,2,3,4}$ can form diagonals of any rectangular, not necessarily a square, figure 16. One can check, that this is indeed a solution by a straightforward repetition of the derivation in section 3: some coefficients at present depend on the angle between $\vec{k}_1$ and $\vec{k}_2$, which are now not orthogonal, but these angles drop out of the final relation (3.27). What makes this deformation interesting is that the change of vectors $\vec{k}_a$ is not a linear transformation in the space of variables $(z, \mathbf{v})$, thus SO$(4,2)$ invariance is not sufficient to explain their existence. Also, the existence of deformed solutions supports the belief that simple solutions with $n > 4$ can exist: rectangular (rather than square) quadruples of $\vec{k}$-vectors naturally arise in degenerations of regular polygons.

Let us summarize the role of symmetries in the space of solutions (3.9) with $n = 4$:

Lorentz symmetry SO$(3,1)$ changes external momenta $\mathbf{p}_a$ and the shape of the non-planar skew quadrilateral in target space. The only invariants are light-likeness of the sides

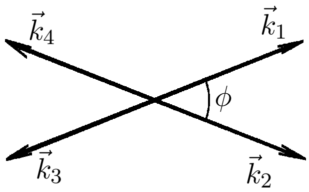

Figure 16: Alternative choice of $\vec{k}$-vectors on the u-plane, which differs from (14) but also provides a solution (3.9) with the *same* boundary conditions to the σ-model equations of motion. The $\vec{k}$-vectors lie along diagonals of a *rectangular* and all have the same length, $\vec{k}_a^2 = L = 2$.

and the lengths $\sqrt{s}$ and $\sqrt{t}$ of the diagonals. The shape of the projection on the (y_1, y_2) plane can be changed from kite or even more generic configurations to a rhombus. This can be considered as a change of coefficients $\mathbf{v}_a$ at fixed z_a.

Conformal symmetry SO(4, 2) allows to make linear transformations of both z_a and $\mathbf{v}_a$ parameters.

None of these target-space symmetries allows to change the vectors $\vec{k}_a$. However, at least for $n = 4$ such a change is possible: all rectangular configurations of four $\vec{k}_a$ provide solutions. This can be considered as a certain rescaling $(u_1, u_2) \to (\alpha u_1, \beta u_2)$, which does not look like an obvious symmetry of the system.

4. Regularized minimal action

After regularizing the action in the way proposed in [1], one has to evaluate an integral of the form

$$\int L_\epsilon \, z^\epsilon d^2u \tag{4.1}$$

where L_ϵ is a certain modification of either the σ-model or the Nambu-Goto action. An attractive feature of this formula is that the finite contribution comes from the *second-order* term in the expansion in powers of ϵ, so that it has a chance to acquire a biliner form, which is needed to reproduce the double contour integral formula (1.9). For the same reason, however, different formulations of the minimal area problem, while equivalent for $\epsilon = 0$, will not necessarily lead to the same answer for ϵ-finite terms in (4.1). In this section we examine the dependence of (4.1) on the choice of z_a and minimize it with respect to them.

4.1 A puzzle

An apparent problem with equation (4.1), arising already in the case of $n = 4$, is the possibility to choose all $z_a = 1$, which makes the answer fully independent of external momenta. To be more precise, the prescription of [1] just to make the only replacement $r \to r\sqrt{1 + \epsilon/2}$ in the solution of equations of motion *at zero* ϵ and insert it back into the action (4.1), implies the substitution

$$\int \frac{(\partial r)^2 + (\partial \mathbf{y})^2}{r^{2+\epsilon}}\, d^2u \longrightarrow \frac{1}{(1+\epsilon/2)^{\epsilon/2}} \int \frac{(1+\epsilon/2)(\partial r)^2 + (\partial \mathbf{y})^2}{(1+\epsilon/2)r^2} \frac{d^2u}{r^\epsilon}$$
$$= \frac{1}{(1+\epsilon/2)^{\epsilon/2}} \int \left(\frac{(\partial r)^2}{r^2} + \frac{1}{1+\epsilon/2} \left(L - \frac{(\partial r)^2}{r^2} \right) \right) \frac{d^2u}{r^\epsilon}$$
$$= \frac{1}{(1+\epsilon/2)^{1+\epsilon/2}} \int \left(L + \frac{\epsilon}{2} \frac{(\partial z)^2}{z^2} \right) z^\epsilon d^2u \tag{4.2}$$

where $(\partial \mathbf{y})^2$ is expressed through r and its derivatives from $L = \frac{(\partial r)^2 + (\partial \mathbf{y})^2}{r^2} = 2$. The factor $(1+\epsilon/2)^{-(1+\epsilon/2)} = 1 - \frac{\epsilon}{2} + O(\epsilon^3)$. Under naive application of the prescription of [1], expression (4.2) is a concrete realization of (4.1), and it suffers from the same problem: it depends only on z and becomes trivial (independent of external momenta) with an allowed choice of all z_a equal. Note that the same argument does not work, at least in such a simple form, for the Nambu-Goto action, because the y-variables do not disappear from the final expression above.

4.2 Equations of motion

We forget for a while the prescription (4.2) and start directly from the deformed σ-model action,

$$\int \frac{(\partial r)^2 + (\partial \mathbf{y})^2}{r^{2+\epsilon}}\, d^2u \tag{4.3}$$

We keep the notation L and z for the old quantities, $L = ((\partial r)^2 + (\partial \mathbf{y})^2)/r^2$ and $z = r^{-1}$, while the new ϵ-dependent quantities will be marked by tildes: $\tilde{L} = ((\partial r)^2 + (\partial \mathbf{y})^2)/r^{2+\epsilon}$ and $\tilde{z} = r^{-1-\epsilon}$.

The equations of motion now read

$$\partial \left(\frac{\partial r}{r^{2+\epsilon}} \right) + \frac{(2+\epsilon)}{2} \frac{L}{r^{1+\epsilon}} = 0,$$
$$\partial \left(\frac{\partial \mathbf{y}}{r^{2+\epsilon}} \right) = 0 \tag{4.4}$$

or, in terms of $\mathbf{v} = \mathbf{y}/r$,

$$\Delta \tilde{z} = (1+\epsilon)\Big(1 + \frac{\epsilon}{2}\Big) \tilde{z} L,$$
$$\partial\Big((1+\epsilon)\tilde{z}\partial \mathbf{v} - \mathbf{v}\partial \tilde{z}\Big) = 0,$$
$$(1+\epsilon)^2 \tilde{z}^2 L = \left\{ (\partial \tilde{z})^2 + \Big((1+\epsilon)\tilde{z}\partial \mathbf{v} - \mathbf{v}\partial \tilde{z}\Big)^2 \right\} \tag{4.5}$$

In contrast to the case of $\epsilon = 0$, the mixing term in the second equation now survives, and the equation for $\mathbf{v}$ does not look like the first equation for $\tilde{z}$. To avoid confusion, we emphasize that in the above equations $L = \tilde{L}_{\epsilon=0}$.

4.3 ϵ-deformed one-cusp solution of [18, 1, 12]

This solution is a direct generalization of (3.62). It possesses an immediate generalization to (3.64), but not to the generic one-cusp limit.

If the only non-vanishing components of z and $\mathbf{v}$ are z_1 and $v^\pm_{\pm 2}$, then one can write

$$\begin{aligned} z &= z_1 E_1, \\ v^\pm &= v^\pm_{\pm 2} E^\gamma_{\pm 2} \end{aligned} \tag{4.6}$$

where γ is some ϵ-dependent power, which still needs to be determined. The crucial problem with the generic single-cusp limit is that, for $\epsilon \neq 0$, one can not safely add the term $\mathbf{v}_1 E_1^\gamma$ to $\mathbf{v}$: with the value of γ required, it then contributes an undesired term $E_{1+1+1+1}$ to the equations. Therefore, we consider restricted ansatz (4.6) without reliable justification (no symmetry transformation is immediately available at $\epsilon \neq 0$ to bring any one-cusp limit to this form).

Substituting (4.6) into (4.5) one obtains

$$\begin{aligned} (1+\epsilon)^2 k_1^2 (z_1 E_1)^{1+\epsilon} &= (1{+}\epsilon)(1{+}\epsilon/2) L (z_1 E_1)^{1+\epsilon}, \\ \Big(\gamma \vec{k}_2 + (1+\epsilon)\vec{k}_1\Big)\Big(\gamma \vec{k}_2 - \vec{k}_1\Big) v_2^+ E_2^\gamma E_1^{1+\epsilon} &= 0, \\ \Big(-\gamma \vec{k}_2 + (1+\epsilon)\vec{k}_1\Big)\Big(-\gamma \vec{k}_2 - \vec{k}_1\Big) v_{-2}^- E_{-2}^\gamma E_1^{1+\epsilon} &= 0, \\ (1{+}\epsilon)^2 (z_1 E_1)^{2(1+\epsilon)} \Big(-L{+}\vec{k}_1^2{+}\Big(\gamma \vec{k}_2{-}\vec{k}_1\Big) v_2^+ \Big(-\gamma \vec{k}_2{-}\vec{k}_1\Big) v_{-2}^-\Big) &= 0 \end{aligned} \tag{4.7}$$

Given $\vec{k}_1^2 = \vec{k}_2^2 = 2$ and $\vec{k}_1 \vec{k}_2 = 0$, these equations imply

$$\begin{aligned} L &= 2\frac{1+\epsilon}{1+\epsilon/2}, \\ \gamma^2 &= 1+\epsilon, \\ L &= 2 + 2(1-\gamma^2) v_2^+ v_2^- = 2 - \epsilon v_2^+ v_2^- \end{aligned} \tag{4.8}$$

or

$$v_2^+ v_2^- = \frac{2}{\epsilon}\left(1 - \frac{1+\epsilon}{1+\epsilon/2}\right) = -\frac{1}{1+\epsilon/2} \tag{4.9}$$

We repeat that this nice exact solution is a deformation of a very special type of a cusp solution at $\epsilon = 0$ — with $\mathbf{v}$ growing slower than z, $\mathbf{v}_1 = 0$ at $z_1 \neq 0$, and it is hard to extract any information from it, which can be justly used in application to solution from more general classes. As we saw, the prescription $r \to r\sqrt{1+\epsilon/2}$ or rather $\mathbf{v} \to \mathbf{v}/\sqrt{1+\epsilon/2}$, while successfully applied in [1], cannot work equally well for all $4d$-equivalent solutions with different sets of $\{z_a\}$.

4.4 Alternative one-cusp solution

If one considers the generic one-cusp limit with $\mathbf{v}$ growing at the same rate as z, then the exponential form of asymptotics (3.9) is no longer true, and this once again demonstrates that the equivalence between different solutions is violated as a result of the ϵ-regularization. This change of asymptotical behavior is a characteristic feature of Whitham deformations: when equations are infinitesimally deformed, the change of solutions is not quite infinitesimal — it is at any given value of the arguments $\vec{u}$, but for a given ϵ and sufficiently large $\vec{k}\vec{u}$ the deformation can be as big as one wishes, the asymptotics is changed, or, in other words, the large $\vec{k}\vec{u}$ and small ϵ limits do not commute.

To be concrete, consider the one-cusp limit $E_b \to \infty$ with all non-vanishing coefficients z_b and $\mathbf{v}_b$. Before ϵ-regularization solution (3.9) in this limit becomes simply

$$\begin{aligned} z &= z_b E_b + O(E_{b,b\pm 1}), \\ \mathbf{v} &= \mathbf{v}_b E_b + O(E_{b,b\pm 1}) \end{aligned} \tag{4.10}$$

Let us simply neglect all $O(E_{b,b\pm 1})$ terms, i.e. demand that z_b and $\mathbf{v}_b$ are the only non-vanishing coefficients. Then

$$z\partial\mathbf{v} - \mathbf{v}\partial z = 0 \tag{4.11}$$

and let us look for a solution to ϵ-deformed equations (4.5) with exactly the same property (4.11). This is not going to be a generic solution, but still it provides some useful information.

Since (4.11) is nothing but

$$(1+\epsilon)\tilde{z}\partial\mathbf{v} - \mathbf{v}\partial\tilde{z} = 0, \tag{4.12}$$

this restriction drastically simplifies (4.5) and reduces it to

$$\begin{aligned} \Delta\tilde{z} &= (1+\epsilon)(1+\epsilon/2)\tilde{z}L, \\ (\partial\tilde{z})^2 &= (1+\epsilon)^2\tilde{z}^2 L \end{aligned} \tag{4.13}$$

It follows that

$$\tilde{z}\Delta\tilde{z} = \frac{1+\epsilon/2}{1+\epsilon}(\partial\tilde{z})^2 \tag{4.14}$$

or

$$\partial(\partial\log z) = \sigma(\partial\log z)^2 \tag{4.15}$$

with $\sigma = -\epsilon/2$ (there would be an additional factor of $(1+\epsilon)^{-1}$ in σ if (4.15) was written in terms of $\tilde{z}$). This equation is easily converted into the Laplace equation

$$\Delta z^{-1/\sigma} = 0 \tag{4.16}$$

with the real part of any complex analytic function as generic solution. Since we are interested in a solution which behaves as $\log z = \vec{k}\vec{u} + O(\sigma)$ as $\sigma \to 0$, one can easily solve (4.15) iteratively and obtain

$$z = z_b\mathcal{E}_b = z_b\frac{1}{(1-\sigma\vec{k}_b\vec{u})^{1/\sigma}} \tag{4.17}$$

The corresponding L is no longer a constant at $\sigma \neq 0$:

$$L = \frac{\vec{k}_b^2}{(1-\sigma\vec{k}_b\vec{u})^2}, \qquad \vec{k}^2 = 2 \tag{4.18}$$

Of course, for a given $\vec{u}$, $\mathcal{E}_b = (1-\sigma\vec{k}_b\vec{u})^{-1/\sigma} \to E_b = e^{\vec{k}_b\vec{u}}$ as $\sigma \to 0$, but at given $\sigma \neq 0$ the asymptotic behavior at $\vec{k}_b\vec{u} \to \infty$ is completely different: $\mathcal{E}_b$ grows/falls faster and reaches infinity/zero at *finite* values of $\vec{k}_b\vec{u} = \pm\sigma^{-1}$. Such a drastic change of the asymptotic behavior is a well-known phenomenon in Whitham theory. As a side remark, note that for infrared regularization ϵ should be negative and $\sigma = -\epsilon/2$ positive.

Unfortunately, this alternative one-cusp solution has the same drawbacks as the previous one: it does not provide us with a complete polygon solution. Indeed, one may try now to substitute (3.9) at $\epsilon \neq 0$ with $\mathcal{E}$ in place of all E. However, such a substitution does not provide an exact solution to equations (4.5). Instead

$$\begin{aligned} z\partial\mathbf{v} - \mathbf{v}\partial z &= \sum_{a,b} k_b \mathcal{P}_{ab}\mathcal{E}_a\mathcal{E}_b', \\ \partial(z\partial\mathbf{v} - \mathbf{v}\partial z) &= \sigma(1+\sigma)\sum_{a,b}\frac{\mathcal{P}_{ab}\left((\vec{k}_a-\vec{k}_b)\vec{u}\right)\left(2-\sigma(\vec{k}_a+\vec{k}_b)\vec{u}\right)}{(1-\sigma\vec{k}_a\vec{u})^{2+1/\sigma}(1-\sigma\vec{k}_b\vec{u})^{2+1/\sigma}} \end{aligned} \tag{4.19}$$

Moreover, L then turns into

$$L = \frac{\sum_{a,b} z_a z_b(\mathcal{E}_a''\mathcal{E}_b + \mathcal{E}_a\mathcal{E}_b'') + \epsilon\vec{k}_a\vec{k}_b\mathcal{E}_a'\mathcal{E}_b'}{(1+\frac{\epsilon}{2})\sum_{a,b} z_a z_b\mathcal{E}_a\mathcal{E}_b} \tag{4.20}$$

and is still independent of the external momenta $\mathbf{p}_a$.

4.5 The σ-model action

We now return to the area integral (4.2) for the solution (3.9) with $n = 4$ and then consider the analogous integral for the deformation of the Nambu-Goto action a la [1].

In order to calculate the regularized σ-model action (4.2), one may use the formula

$$\begin{aligned} \mathcal{R}_\epsilon &\equiv \frac{1}{(1+\epsilon/2)^{1+\epsilon/2}}\int\left(L + \frac{\epsilon}{2}\frac{(\partial z)^2}{z^2}\right)z^\epsilon d^2u \\ &= \frac{1}{(1+\epsilon/2)^{1+\epsilon/2}}\left(2 - \frac{1}{2(1-\epsilon)}\sum_{a,b}(\vec{k}_a\vec{k}_b)z_a z_b\frac{\partial^2}{\partial z_a\partial z_b}\right)\int z^\epsilon d^2u \end{aligned} \tag{4.21}$$

derived in appendix A. In [1] the same quantity was parameterized by two functions I_1 and I_2

$$\mathcal{R}_\epsilon \;=\; 2\int z^\epsilon(1+\epsilon I_1 + \epsilon^2 I_2 + \ldots)d^2u \tag{4.22}$$

According to (4.21), integrals with I_1 and I_2 will be immediately known (note that they themselves depend on ϵ!), once one evaluates

$$\mathcal{J}\{z_a\} = \int z^\epsilon d^2u \tag{4.23}$$

This integral is calculated in appendix A (the calculation generalizes the calculation of [1, appendix B] to arbitrary values of z_a). The answer is (see appendix A for details)

$$\mathcal{R}_\epsilon = \mathcal{K}_\epsilon \left\{ 1 + \frac{\epsilon}{4}\log(z_1 z_2 z_3 z_4) + \frac{\epsilon^2}{8}\log(z_1 z_3)\log(z_2 z_4) \right\} \tag{4.24}$$

with

$$\mathcal{K}_\epsilon = \tilde{K}_\epsilon \left(1 + \frac{\epsilon}{2} + \frac{\epsilon^2}{2} \right) = \frac{8}{\epsilon^2 |\sin\phi|}\left(1 + \epsilon^2 \left(\frac{1}{4} - \frac{\pi^2}{12} \right) \right) \tag{4.25}$$

where ϕ is the angle between the vectors $\vec{k}_1$ and $\vec{k}_2$ (we consider here the general case of a rectangle). Eq. (4.24) is our final formula for the regularized "minimal area" for $n = 4$ in the σ-model approach. Taking in addition into account the constraint (3.27), one obtains a non-trivial function of the kinematical variables s, t and either one of the products $z_1 z_3$ and $z_2 z_4$.

For the Alday-Maldacena choice $z_1 = z_3 = 1 - b$ and $z_2 = z_4 = 1 + b$ and $\phi = \frac{\pi}{2}$ one obtains from (4.24):

$$\mathcal{R}_\epsilon^{AM} = \frac{1}{2}\mathcal{K}_\epsilon \left\{ (1-b)^\epsilon + (1+b)^\epsilon - \frac{\epsilon^2}{2}\left(\log\frac{1-b}{1+b} \right)^2 \right\} \tag{4.26}$$

According to [1] this expression should be multiplied by

$$\frac{\sqrt{\lambda_D c_D}}{2\pi a^\epsilon} = \frac{\sqrt{\lambda \mu^{2\epsilon}}(2\pi)^\epsilon \sqrt{1+\epsilon}}{2\pi a^\epsilon \sqrt{1 - \frac{\pi^2 \epsilon^2}{12}}} \tag{4.27}$$

to give

$$\mathrm{Area}_\epsilon = 2^{1+2\epsilon}\frac{\tilde{\mathcal{K}}_\epsilon}{\pi\epsilon^2}\left\{ \sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} + \sqrt{\frac{\lambda\mu^{2\epsilon}}{(-t)^\epsilon}} - \frac{\epsilon^2}{8}\left(\log\frac{s}{t} \right)^2 \right\} \tag{4.28}$$

where

$$\tilde{\mathcal{K}}_\epsilon = 1 + \frac{\epsilon}{2}(1 - \log 2) + \frac{\epsilon^2}{8}\left(1 - \frac{\pi^2}{3} - 2\log 2 + (\log 2)^2 \right) \tag{4.29}$$

Note that our normalization of z contains an extra factor of 4 as compared to [1] (the sum of four exponents is 4 times a product of two cosines) and this contributes a factor 4^ϵ in the answer. An extra factor of 2 is due to our choice of the constant $L = 2$, instead of the $L = 1$ of [1].

Thus, (4.24) is in full agreement with [1], but one can use it equally well for other choices of $\{z_a\}$, and the answer obviously depends on this choice. Even more striking, it strongly depends on the angle ϕ between the vectors $\vec{k}_1$ and $\vec{k}_2$, which, along with $\{z_a\}$, are the moduli in the solution space.

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4.6 The Alday-Maldacena solution as a minimum in the moduli space

Since solutions with different $\{z_a\}$ are not equivalent, the regularized action depends on these parameters. In such a case, one should naturally look for a new extremum: the minimum of (4.24) in the moduli space of solutions. Since the moduli space is the hyper-surface (3.27)

$$z_1 z_3 s + z_2 z_4 t = 1 \tag{4.30}$$

in the space of z_a-variables, we should look for a minimum of (4.24) with respect to z_a under the constraint (4.30). This gives

$$\begin{aligned}
\frac{1}{\epsilon z_1}\Big(1+\frac{\epsilon}{2}\log(z_2 z_4)\Big) &= \lambda z_3 s,\\
\frac{1}{\epsilon z_2}\Big(1+\frac{\epsilon}{2}\log(z_1 z_3)\Big) &= \lambda z_4 t,\\
\frac{1}{\epsilon z_3}\Big(1+\frac{\epsilon}{2}\log(z_2 z_4)\Big) &= \lambda z_1 s,\\
\frac{1}{\epsilon z_4}\Big(1+\frac{\epsilon}{2}\log(z_1 z_3)\Big) &= \lambda z_2 t,
\end{aligned} \tag{4.31}$$

where λ is the Lagrange multiplier. A possible solution of this system is:

$$\begin{aligned}
z_1 = z_3 &= \frac{1}{\sqrt{2s}}\left(1-\frac{\epsilon}{8}\log\frac{s}{t}+O(\epsilon^2)\right),\\
z_2 = z_4 &= \frac{1}{\sqrt{2t}}\left(1+\frac{\epsilon}{8}\log\frac{s}{t}+O(\epsilon^2)\right)
\end{aligned} \tag{4.32}$$

i.e. we obtain exactly the Alday-Maldacena choice (3.35) for z_a, with corrections of order ϵ. The latter could lead to a different finite term in (4.24), however they exactly cancel each other, since as one may check

$$\frac{\epsilon}{4}\log(z_1 z_2 z_3 z_4) = -\frac{\epsilon}{4}\log(4st) + O(\epsilon^3) \tag{4.33}$$

Of course, (4.32) is not the generic solution of (4.31). In fact, from (4.24) it follows that the ϵ-regularization did not fully break the non-Lorentz part of SO(4, 2): the answer depends on the products $z_1 z_3$ and $z_2 z_4$ and is invariant under rescalings $z_1 \to \alpha z_1$, $z_3 \to z_3/\alpha$ and $z_2 \to \beta z_1$, $z_4 \to z_4/\beta$. However, exactly because (4.24) depends only on the two products, the other solutions, obtained by such rescalings, do not affect the vanishing of the correction in (4.33).

The regularized action (4.24) depends also on the angle ϕ through a factor $|\sin\phi|^{-1}$ in $\mathcal{K}_\epsilon$, and minimum is obviously located at $\phi = \pi/2$.

4.7 The Nambu-Goto action

Nambu-Goto (NG) action seems a more difficult issue to address in the approach of [1], though it is much better from geometrical and, perhaps, even conceptual points of view [33].

First of all, as mentioned in section 3.1.3, the NG Lagrangian is not invariant under T-duality. This means that one needs to borrow the beautiful formulation of the minimal area problem with boundary formed by external momenta from the σ-model formalism and then use it as a starting point for NG calculations. In principle, this is not a drawback, especially if one believes that the origin of the AdS/CFT (string/gauge) duality is in Polyakov's formalism with the Liouville field playing the role of the $5-th$ dimension: then the σ-model-like formulation is the starting point in any case and the NG-like formulation in terms of minimal areas is a derivable concept.

Second, not all the σ-model solutions (3.9) necessarily satisfy the Nambu-Goto equations of motion (Alday-Maldacena solution does). Description of moduli space of Nambu-Goto solutions is an open problem even for $n=4$.

Third, if this moduli space is also large, as in the σ-model case, further calculations can be more difficult. For a given solution one can not immediately get rid of the y-fields when the action is evaluated with the help of the Alday-Maldacena prescription (to substitute r by $r\sqrt{1+\epsilon/2}$), as we did in (4.2): some direct calculation involving all the fields z and v should be performed. It is an intriguing question, if the result will be the same as in the σ-model case, and — even if not — if the Alday-Maldacena solution is still a minimum in the moduli space.

5. Conclusion and prospects

To summarize, the seemingly *ad hoc* choice of z_a-parameter values made in [1] indeed corresponds to a minimum of the regularized area in the moduli space of all possible solutions of the AdS σ-model with given boundary conditions and the one-to-one correspondence between minimal surfaces and boundary conditions is partly restored after regularization. Of course, whether or not the minimal surface is regularization independent remains an open question. In general, in order to find the minimum one needs to know the area as a function on the moduli space, i.e. analysis of some particular solution is not sufficient. In practice, however, it appeared sufficient to restrict consideration to the SO$(4,2)$-orbit of a particular solution, despite the fact that this symmetry does not act transitively on the entire moduli space (for instance, it does not change the angle ϕ between the $\vec{k}$-vectors). Since the SO$(4,2)$ symmetry is broken by regularization, the z_a-dependence of the regularized action is not automatically given by symmetry arguments and should be determined by straightforward calculation. This implies that to address the $n>4$ problem one will need to construct the full family of solutions and evaluate their regularized action.

Modulo the above comments, the reasoning of [1] reduces in the framework of the AdS/CFT correspondence the problem of n-point amplitudes of $N=4$ SYM at strong coupling to a couple of well-defined problems in the field of integrable systems:

(i) Find solutions of the $2d$ integrable SO$(4,2)$ sigma-model allowing for growing asymptotics on the world sheet, and

(ii) Find their Whitham deformations, induced by an ϵ-regularization, which breaks the integrability of the σ-model.

The regularized minimal area is then defined by a minimum of some still-to-be-determined function on the moduli space of solutions.

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Using the results of [9] one may write the BDS conjectured formula, given in the introduction, in terms of the regularized Polyakov's [22] double contour integral (1.9) over the auxiliary polygon Π in momentum space, formed by the momenta of the scattering process under study, i.e.

$$\text{Amplitude in perturbative } N = 4 \text{ SYM} \quad \sim \quad \exp\left(\frac{\gamma(\lambda)}{4}\oint_{\Pi}\oint_{\Pi}\frac{dy^{\mu}dy'_{\mu}}{(y-y')^{2+\epsilon}}\right), \tag{5.1}$$

Then the AdS/CFT duality in the sector of n-point amplitudes, may equivalently be stated in purely *geometrical* terms, namely: why does this integral coincide with the area of the minimal surface Σ defined by above Whitham-deformed σ-model solutions:

$$\oint_{\Pi}\oint_{\Pi}\frac{dy^{\mu}dy'_{\mu}}{(y-y')^{2+\epsilon}} = \text{Area}_{\epsilon}(\Sigma), \qquad \Pi = \partial\Sigma \tag{5.2}$$

This was explicitly verified in [1] and also here for $n = 4$. The exact relation between integrable structures on the two sides of this formula [25, 43, 44] remains to be understood. We emphasize that the result of [1] is entirely based on the *deviations* from ordinary integrability. It suggests the crucial role of a very different — *Whitham* – integrability [36, 37], which did not yet attract much attention in the studies of the AdS/CFT correspondence.

A. Evaluation of the regularized area

In this appendix, we explain how to calculate the regularized area in the σ-model case (4.2). The calculation is a straightforward extension of the calculation in [1] to the case of generic z_a.

For $z = \sum_a z_a e^{\vec{k}_a\vec{u}}$ we have:

$$\left(\frac{\partial}{\partial\vec{u}} - \sum_a \vec{k}_a z_a \frac{\partial}{\partial z_a}\right) z = 0 \quad \Longrightarrow \quad \frac{(\partial z)^2}{z^2} = \sum_{a,b}(\vec{k}_a\vec{k}_b)\frac{z_a z_b}{z^2}\frac{\partial z}{\partial z_a}\frac{\partial z}{\partial z_b} \tag{A.1}$$

Next, from

$$\frac{\partial^2 z}{\partial z_a \partial z_b} = 0 \quad \Longrightarrow \quad \frac{\partial^2 z^{\epsilon}}{\partial z_a \partial z_b} = \epsilon(\epsilon-1)\frac{z^{\epsilon}}{z^2}\frac{\partial z}{\partial z_a}\frac{\partial z}{\partial z_b} \tag{A.2}$$

so that

$$\frac{(\partial z)^2}{z^2}\, z^{\epsilon} = -\frac{1}{\epsilon(1-\epsilon)}\sum_{a,b}(\vec{k}_a\vec{k}_b) z_a z_b \frac{\partial^2 z^{\epsilon}}{\partial z_a \partial z_b} \tag{A.3}$$

Thus substituting (3.9) into (4.2) we obtain:

$$\begin{aligned} \mathcal{R}_{\epsilon} &\equiv \frac{1}{(1+\epsilon/2)^{1+\epsilon/2}}\int\left(L + \frac{\epsilon}{2}\frac{(\partial z)^2}{z^2}\right) z^{\epsilon} d^2u \\ &= \frac{1}{(1+\epsilon/2)^{1+\epsilon/2}}\left(2 - \frac{1}{2(1-\epsilon)}\sum_{a,b}(\vec{k}_a\vec{k}_b) z_a z_b \frac{\partial^2}{\partial z_a \partial z_b}\right)\int z^{\epsilon} d^2u \end{aligned} \tag{A.4}$$

In [1] the same quantity was parameterized by two functions I_1 and I_2:

$$\mathcal{R}_\epsilon \ = \ 2\int z^\epsilon(1+\epsilon I_1+\epsilon^2 I_2+\ldots)d^2u \tag{A.5}$$

and looking at the l.h.s. of (A.4) with $L=2$ and $(1+\epsilon/2)^{-(1+\epsilon/2)} = 1-\epsilon/2+O(\epsilon^3)$, one immediately sees that

$$2I_2+I_1 = -\frac{1}{2} \tag{A.6}$$

According to (A.4) integrals with I_1 and I_2 will be immediately known (note that they themselves depend on ϵ!), if one evaluates

$$\mathcal{J}_n\{z_a\} = \int z^\epsilon d^2u \tag{A.7}$$

If $n=4$ and $\vec{k}_3 = -\vec{k}_1$ and $\vec{k}_4 = -\vec{k}_2$, as is the case for our solutions, then we can use $\tilde{u}_1 = \vec{k}_1\vec{u}$ and $\tilde{u}_2 = \vec{k}_2\vec{u}$ as new coordinates on the world sheet, and they can be further shifted by $\frac{1}{2}\log(z_1/z_3)$ and $\frac{1}{2}\log(z/2/z_4)$ respectively in order to give:

$$\begin{aligned}\mathcal{J}_4\{z_a\} &= \frac{2^\epsilon}{|\vec{k}_1\times\vec{k}_2|}\int\Big(\sqrt{z_1z_3}\cosh\tilde{u}_1+\sqrt{z_2z_4}\cosh\tilde{u}_2\Big)^\epsilon d^2\tilde{u}\\ &= \frac{2^{1+\epsilon}}{|\vec{k}_1\times\vec{k}_2|}\int\Big\{\Big(\sqrt{z_1z_3}+\sqrt{z_2z_4}\Big)\cosh\hat{u}_1\cosh\hat{u}_2\\ &\qquad\qquad +\Big(\sqrt{z_1z_3}-\sqrt{z_2z_4}\Big)\sinh\hat{u}_1\sinh\hat{u}_2\Big\}^\epsilon d^2\hat{u}\end{aligned} \tag{A.8}$$

In the second line the variables are rotated once again, $\hat{u}_1 = \frac{\tilde{u}_1+\tilde{u}_2}{2}$, $\hat{u}_2 = \frac{\tilde{u}_1-\tilde{u}_2}{2}$, and the resulting integral can be evaluated, say, by expanding in powers of the second item, as was done in [1]:

$$\begin{aligned}&\int\Big(A\cosh\hat{u}_1\cosh\hat{u}_2+B\sinh\hat{u}_1\sinh\hat{u}_2\Big)^\epsilon d^2\hat{u}\\ &\quad=\frac{A^\epsilon}{\Gamma(-\epsilon)}\sum_{k=0}^\infty\frac{\Gamma(2k-\epsilon)}{(2k)!}\left(-\frac{B}{A}\right)^{2k}\left(\int\tanh^{2k}\hat{u}\cosh^\epsilon\hat{u}\,d\hat{u}\right)^2\end{aligned} \tag{A.9}$$

The last integral converges for $\epsilon<0$ and is given by a B-function formula with $\xi=\tanh\hat{u}$:

$$\int\tanh^{2k}\hat{u}\cosh^\epsilon\hat{u}\,d\hat{u} = \int_0^1(\xi^2)^{k-1/2}(1-\xi^2)^{-1-\epsilon/2}d(\xi^2) = \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma\left(k+\frac{1-\epsilon}{2}\right)} \tag{A.10}$$

It remains to substitute (A.10) into (A.9) and make use of the doubling formula

$$\Gamma(2k-\epsilon) = \frac{2^{2k-\epsilon-1}}{\sqrt{\pi}}\Gamma\left(k-\frac{\epsilon}{2}\right)\Gamma\left(k+\frac{1-\epsilon}{2}\right) \tag{A.11}$$

to obtain [1]:

$$\begin{aligned}(A.9) &= \frac{A^\epsilon\Gamma^2(-\frac{\epsilon}{2})}{2^{1+\epsilon}\Gamma(-\epsilon)}\sum_{k=0}^\infty\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k-\frac{\epsilon}{2}\right)}{k!\;\Gamma\left(k+\frac{1-\epsilon}{2}\right)}\left(-\frac{B}{A}\right)^{2k}\\ &= \frac{\pi A^\epsilon\Gamma^2\big(-\frac{\epsilon}{2}\big)}{\Gamma^2\big(\frac{1-\epsilon}{2}\big)}\;{}_2F_1\left(\frac{1}{2},-\frac{\epsilon}{2};\frac{1-\epsilon}{2};\left(\frac{B}{A}\right)^2\right)\end{aligned} \tag{A.12}$$

At the last step one applies doubling to $\Gamma(-\epsilon)$ and includes a factor $\frac{\Gamma(\frac{1}{2})\Gamma(-\frac{\epsilon}{2})}{\Gamma(\frac{1-\epsilon}{2})}$ from the definition of the F-function.

Now we return to (A.4) and substitute the evaluated expression (A.12) for $\mathcal{J}_4$. The differential equation for the hypergeometric function ${}_2F_1$ can be used to evaluate the derivatives. Alternatively, one can expand (A.12) and keep the first relevant powers of ϵ already at this stage, as was done in [1] with the help of the asymptotic formula

$$ {}_2F_1\left(\frac{1}{2},-\frac{\epsilon}{2};\frac{1-\epsilon}{2};C\right)=1+\frac{\epsilon}{2}\log(1-C)+\frac{\epsilon^2}{2}\log(1-\sqrt{C})\log(1+\sqrt{C})+O(\epsilon^3) \tag{A.13} $$

so that

$$ \begin{aligned} &\left(1+\frac{\epsilon}{2}\log(A^2-B^2)+\frac{\epsilon^2}{2}\log(A-B)\log(A+B)+O(\epsilon^3)\right)\\ &\quad=K_\epsilon\left(1+\frac{\epsilon}{4}\log(16z_1z_2z_3z_4)+\frac{\epsilon^2}{8}\log(4z_1z_3)\log(4z_2z_4)+O(\epsilon^3)\right)\\ &\quad=2^\epsilon K_\epsilon\left(1+\frac{\epsilon}{4}\log(z_1z_2z_3z_4)+\frac{\epsilon^2}{8}\log(z_1z_3)\log(z_2z_4)+O(\epsilon^3)\right)\\ &\quad=2^\epsilon K_\epsilon\left\{1+\frac{\epsilon}{4}\log(z_1z_2z_3z_4)+\frac{\epsilon^2}{32}\Big(\log(z_1z_2z_3z_4)\Big)^2-\frac{\epsilon^2}{32}\Big(\log\frac{z_1z_3}{z_2z_4}\Big)^2+O(\epsilon^3)\right\} \end{aligned} \tag{A.14} $$

where

$$ \begin{aligned} K_\epsilon&=\frac{\pi\Gamma^2\big(-\frac{\epsilon}{2}\big)}{\Gamma^2\big(\frac{1-\epsilon}{2}\big)}=\frac{4^{1-\epsilon}}{\epsilon^2}\left(\frac{\Gamma^2(1-\frac{\epsilon}{2})}{\Gamma(1-\epsilon)}\right)^2=\frac{4^{1-\epsilon}}{\epsilon^2}\left(1+\frac{\epsilon^2}{2}\Big[\big(\Gamma'(1)\big)^2-\Gamma''(1)\Big]+O(\epsilon^3)\right)\\ &=\frac{4^{1-\epsilon}(1-\frac{\pi^2\epsilon^2}{12})}{\epsilon^2} \end{aligned} \tag{A.15} $$

because $\Gamma(1+z)=1-\gamma z+\left(\frac{\pi^2}{12}+\frac{\gamma^2}{2}\right)z^2+\dots$

If the angle between vectors $\vec{k}_1$ and $\vec{k}_2$ is ϕ, then we obtain, up to the terms of order $O(\epsilon)$

$$ \mathcal{R}_\epsilon=\tilde{K}_\epsilon\left(1-\frac{1}{4(1-\epsilon)}\sum_{a,b}(\vec{k}_a\vec{k}_b)z_az_b\frac{\partial^2}{\partial z_a\partial z_b}\right)\left(1+\frac{\epsilon}{4}\log(z_1z_2z_3z_4)+\frac{\epsilon^2}{8}\log(z_1z_3)\log(z_2z_4)\right) \tag{A.16} $$

with

$$ \tilde{K}_\epsilon=\left(1-\frac{\epsilon}{2}\right)\frac{2^{1+\epsilon}}{2|\sin\phi|}\cdot 2^\epsilon K_\epsilon\cdot 2=\frac{8\left(1-\frac{\epsilon}{2}-\frac{\pi^2\epsilon^2}{12}\right)}{\epsilon^2|\sin\phi|} \tag{A.17} $$

where the last factor 2 comes from $L=2$.

The action of the differential operator on the first logarithmic term is simple, since this logarithm is just a sum $\sum_a\log z_a$, only four terms with $a=b$ and $\vec{k}_a^2=2$ contribute:

$$ \sum_{a,b}(\vec{k}_a\vec{k}_b)z_az_b\frac{\partial^2}{\partial z_a\partial z_b}\log(z_1z_2z_3z_4)=-2\cdot(1+1+1+1)=-8 \tag{A.18} $$

The action on the second logarithmic term is more interesting. For the same reason $\partial^2/\partial z_1\partial z_3$ and $\partial^2/\partial z_2\partial z_4$ annihilate this term, while the four operators $\partial^2/\partial z_a\partial z_{a+1}$ are multiplied by $(\vec{k}_a\vec{k}_{a+1}) = -2(-)^a\cos\phi$ so that they do not contribute at all when $\vec{k}_a$ are directed along diagonals of a *square* and cancel among each other even if $\phi \neq \pi/2$. Keeping this in mind we obtain:

$$
\begin{aligned}
&\sum_{a,b}(\vec{k}_a\vec{k}_b)z_az_b\frac{\partial^2}{\partial z_a\partial z_b}\log(z_1z_3)\log(z_2z_4) \\
&\qquad\qquad = 2(-1-1)\Big(\log(z_2z_4)+\log(z_1z_3)\Big)+4\cos\phi\Big(1-1+1-1\Big) \\
&\qquad\qquad = -4\log(z_1z_2z_3z_4)
\end{aligned}
\tag{A.19}
$$

Therefore (A.16) becomes

$$
\begin{aligned}
\mathcal{R}_\epsilon &= \tilde{K}_\epsilon\left\{\left(1+\frac{\epsilon}{4}\log(z_1z_2z_3z_4)+\frac{\epsilon^2}{8}\log(z_1z_3)\log(z_2z_4)\right)+\frac{8\epsilon}{16(1-\epsilon)}+\frac{4\epsilon^2}{32}\log(z_1z_2z_3z_4)\right\} \\
&= \mathcal{K}_\epsilon\left\{1+\frac{\epsilon}{4}\log(z_1z_2z_3z_4)+\frac{\epsilon^2}{8}\log(z_1z_3)\log(z_2z_4)\right\}
\end{aligned}
\tag{A.20}
$$

with

$$
\mathcal{K}_\epsilon = \tilde{K}_\epsilon\left(1+\frac{\epsilon}{2}+\frac{\epsilon^2}{2}\right) = \frac{8\left(1-\frac{\epsilon}{2}-\frac{\pi^2\epsilon^2}{12}\right)}{\epsilon^2\sin\phi}\left(1+\frac{\epsilon}{2}+\frac{\epsilon^2}{2}\right) = \frac{8}{\epsilon^2|\sin\phi|}\left(1+\epsilon^2\Big(\frac{1}{4}-\frac{\pi^2}{12}\Big)\right)
\tag{A.21}
$$

Equation (A.20) is our final formula for the regularized "minimal area" for $n = 4$ in the σ-model approach.

For the Alday-Maldacena choice of z_a variables $z_1 = z_3 = 1-b$ and $z_2 = z_4 = 1+b$ and $\phi = \frac{\pi}{2}$ we obtain from (A.20):

$$
\begin{aligned}
\mathcal{R}_\epsilon &= \mathcal{K}_\epsilon\left\{1+\frac{\epsilon}{2}\log(1-b^2)+\frac{\epsilon^2}{4}\left[\Big(\log(1-b)\Big)^2+\Big(\log(1+b)\Big)^2-\Big(\log\frac{1-b}{1+b}\Big)^2\right]\right\} \\
&= \frac{1}{2}\mathcal{K}_\epsilon\left\{(1-b)^\epsilon+(1+b)^\epsilon-\frac{\epsilon^2}{2}\Big(\log\frac{1-b}{1+b}\Big)^2\right\}
\end{aligned}
\tag{A.22}
$$

According to [1] this expression should be multiplied by

$$
\frac{\sqrt{\lambda_D c_D}}{2\pi a^\epsilon} = \frac{\sqrt{\lambda\mu^{2\epsilon}}(2\pi)^\epsilon\sqrt{1+\epsilon}}{2\pi a^\epsilon\sqrt{1-\frac{\pi^2\epsilon^2}{12}}}
\tag{A.23}
$$

to give:

$$
\begin{aligned}
\mathrm{Area}_\epsilon &= 2^{1+2\epsilon}\frac{\tilde{\mathcal{K}}_\epsilon}{\pi\epsilon^2}\sqrt{\lambda\mu^{2\epsilon}}\left\{\left[\frac{2\pi(1-b)}{2^{3/2}\,a}\right]^\epsilon+\left[\frac{2\pi(1+b)}{2^{3/2}\,a}\right]^\epsilon-\frac{\epsilon^2}{2}\Big(\log\frac{1-b}{1+b}\Big)^2\right\} = \\
&= 2^{1+2\epsilon}\frac{\tilde{\mathcal{K}}_\epsilon}{\pi\epsilon^2}\left\{\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}}+\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-t)^\epsilon}}-\frac{\epsilon^2}{8}\left(\log\frac{s}{t}\right)^2\right\}
\end{aligned}
\tag{A.24}
$$

where

$$\tilde{\mathcal{K}}_\epsilon = 2^{-\epsilon/2}\left(1+\epsilon^2\Big(\frac{1}{4}-\frac{\pi^2}{12}\Big)\right)\sqrt{\frac{1+\epsilon}{1-\frac{\pi^2\epsilon^2}{12}}}$$
$$= 1+\frac{\epsilon}{2}(1-\log 2)+\frac{\epsilon^2}{8}\left(1-\frac{\pi^2}{3}-2\log 2+(\log 2)^2\right) \tag{A.25}$$

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